First-Order Logic
Herbrand Theory

Herbrand universe

The Herbrand universe $T(F)$ of a closed formula F in Skolem form is the set of all terms that can be constructed using the function symbols in F (including the constants!).

In the special case that F contains no constants, we first pick an arbitrary constant, say a, and then construct the terms.

Formally, $T(F)$ is inductively defined as follows:

- All constants occurring in F belong to $T(F)$; if no constant occurs in F, then $a \in T(F)$ for an arbitrary constant a.
- For every n-ary function symbol f occurring in F, if $t_1, t_2, \ldots, t_n \in T(F)$ then $f(t_1, t_2, \ldots, t_n) \in T(F)$.

Note: All terms in $T(F)$ are variable-free by construction!

Example

$T(\,\forall x\,\forall y\, P(f(x), g(c, y))\,) = \{c, f(c), g(c, c), f(g(c, c)), \ldots\}$.

Herbrand structure

Let F be a closed formula in Skolem form. A structure $\mathcal{A}$ suitable for F is a Herbrand structure for F if it satisfies the following conditions:

- $U^{\mathcal{A}} = T(F)$, and
- for every n-ary function symbol f occurring in F and every $t_1, \ldots, t_n \in T(F)$: $f^{\mathcal{A}}(t_1, \ldots, t_n) = f(t_1, \ldots, t_n)$.

Fact

If $\mathcal{A}$ is a Herbrand structure, then $\mathcal{A}(t) = t$ for all $t \in U^{\mathcal{A}}$.

A Herbrand model of F is a Herbrand structure suitable for F that is model of F.

Matrix of a formula

Definition

The matrix of a formula F is the result of removing all quantifiers (all $\forall x$ and $\exists x$) from F. The matrix is denoted by F^*.

Fundamental theorem of predicate logic

Theorem

A closed formula in Skolem form is satisfiable iff it has a Herbrand model.

Proof ($\Leftarrow$): If a formula has a model then it is satisfiable.

($\Rightarrow$): Let $\mathcal{A}$ be a model of a closed formula F in Skolem form. We define a Herbrand structure $\mathcal{T}$ suitable for F:

Universe:	$U_{\mathcal{T}} = T(F)$
Function symbols:	$f^{\mathcal{T}}(t_1, \ldots, t_n) = f(t_1, \ldots, t_n)$
	(If F contains no constant, then $a^{\mathcal{A}} = u$ for some arbitrary $u \in U^{\mathcal{A}}$)
Predicate symbols:	$(t_1, \ldots, t_n) \in P^{\mathcal{T}}$ iff $(\mathcal{A}(t_1), \ldots, \mathcal{A}(t_n)) \in P^{\mathcal{A}}$

Claim: $\mathcal{T}$ is also a model of F.

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We prove a stronger assertion:

> *For every closed formula G in Skolem form that contains the same function and predicate symbols as F, if $\mathcal{A} \models G$ then $\mathcal{T} \models G$*

Proof By induction on the number n of universal quantifiers of G.

Basis: $n = 0$. Then G has no quantifiers at all.
Hence, G is a boolean combination of atomic formulas without variables.
So $\mathcal{A}(G) = \mathcal{T}(G)$ (why?), and we are done.

Step: $n > 0$. Let $G = \forall x\, H$.

$$\begin{array}{lll}
 & \mathcal{A} \models G & \\
\Rightarrow & \text{for every } u \in U^{\mathcal{A}}\text{: } \mathcal{A}[u/x](H) = 1 & \\
\Rightarrow & \text{for every } u \in U^{\mathcal{A}} \text{ s.t. } u = \mathcal{A}(t) & \\
 & \text{for some } t \in T(F)\text{: } \mathcal{A}[u/x](H) = 1 & \\
\Rightarrow & \text{for every } t \in T(F)\text{: } \mathcal{A}[\mathcal{A}(t)/x](H) = 1 & \\
\Rightarrow & \text{for every } t \in T(F)\text{: } \mathcal{A}(H[t/x]) = 1 & \text{(Subst. Lemma)} \\
\Rightarrow & \text{for every } t \in T(F)\text{: } \mathcal{T}(H[t/x]) = 1 & \text{(IH)} \\
\Rightarrow & \text{for every } t \in T(F)\text{: } \mathcal{T}[\mathcal{T}(t)/x](H) = 1 & \text{(Subst. Lemma)} \\
\Rightarrow & \text{for every } t \in T(F)\text{: } \mathcal{T}[t/x](H) = 1 & (\mathcal{T} \text{ is Herbrand struct.}) \\
\Rightarrow & \mathcal{T}(\forall x\ H) = 1 & (U^{\mathcal{T}} = T(F)) \\
\Rightarrow & \mathcal{T} \models G &
\end{array}$$

Example

Let F and $\mathcal{A}$ be given by

$$\begin{aligned} F &= \forall x\,(x > \mathbf{0} \rightarrow \exists x\,(x > y \wedge y > \mathbf{0})) \\ \mathcal{U}^{\mathcal{A}} &= \mathbb{Q} \\ \mathbf{0}^{\mathcal{A}} &= 0 \\ p >^{\mathcal{A}} q &\Leftrightarrow p > q \end{aligned}$$

$\mathcal{A}$ is a model of F. The Skolem form of F is

$$G = \forall x\,(x > \mathbf{0} \rightarrow (x > f(x) \wedge f(x) > \mathbf{0}))\ .$$

Extending $\mathcal{A}$ with e.g. $f^{\mathcal{A}}(p) = p/2$ makes $\mathcal{A}$ a model of G.

Which is the Herbrand structure $\mathcal{T}$ given by the proof of the fundamental theorem?

Example

The Herbrand structure $\mathcal{T}$ is given by:

$$\begin{aligned}
\mathcal{U}^{\mathcal{T}} &= T(G) = \{\mathbf{0}, f(\mathbf{0}), f(f(\mathbf{0})), \ldots\} = \{f^k(\mathbf{0}) \mid k \geq 0\} \\
f^{\mathcal{T}}(f^k(\mathbf{0})) &= f(f^k(\mathbf{0})) = f^{k+1}(\mathbf{0}) \\
f^k(\mathbf{0}) >^{\mathcal{T}} f^{\ell}(\mathbf{0}) &\Leftrightarrow (f^k(\mathbf{0}))^{\mathcal{A}} >^{\mathcal{A}} (f^{\ell}(\mathbf{0}))^{\mathcal{A}} \\
&\Leftrightarrow (f^{\mathcal{A}})^k(\mathbf{0}^{\mathcal{A}}) >^{\mathcal{A}} (f^{\mathcal{A}})^{\ell}(\mathbf{0}^{\mathcal{A}}) \\
&\Leftrightarrow (f^{\mathcal{A}})^k(0) > (f^{\mathcal{A}})^{\ell}(0) \\
&\Leftrightarrow 0/2^k > 0/2^{\ell} \\
&\Leftrightarrow \text{false}
\end{aligned}$$

The theorem guarantees that $\mathcal{T}$ is also a model of G. This is indeed the case because the premise $x > \mathbf{0}$ of the implication is always false.

We have just shown:

Theorem

Let F be a closed formula in Skolem form.
Then F is satisfiable iff it has a Herbrand model.

What goes wrong if F is not closed or not in Skolem form?

Herbrand expansion

Let $F = \forall y_1 \ldots \forall y_n F^*$ be a closed formula in Skolem form.
The Herbrand expansion of F is the set of formulas

$$E(F) = \{F^*[t_1/y_1] \ldots [t_n/y_n] \mid t_1, \ldots, t_n \in T(F)\}$$

Informally: the formulas of $E(F)$ are the result of substituting terms from $T(F)$ for the variables of F^* in every possible way.

Example

Some elements of $E(\forall x \, \forall y \, P(f(x), g(c, y))$:

$P(f(c), g(c, c))$ $\quad$ $P(f^2(c), g(c, c))$ $\quad$ $P(f(c), g(c, f(c)))$

$P(f^8(c), g(c, c))$ $\quad$ $P(f(g(f(c), f(c))), g(c, f(g(c, f(c)))))$

Note: The Herbrand expansion can be viewed as a set of propositional formulas over the set of variable-free atomic formulas.

Gödel-Herbrand-Skolem Theorem

Theorem

A closed formula F in Skolem form is satisfiable iff its Herbrand expansion $E(F)$ is satisfiable (in the sense of propositional logic).

Proof. By the fundamental theorem, it suffices to show that F has a Herbrand model iff $E(F)$ is satisfiable.

Let $F = \forall y_1 \dots \forall y_n\, F^*$.

$\mathcal{A}$ is a Herbrand model of F

iff for all $t_1, \dots, t_n \in T(F)$, $\mathcal{A}[t_1/y_1]\dots[t_n/y_n](F^*) = 1$

iff for all $t_1, \dots, t_n \in T(F)$, $\mathcal{A}(F^*[t_1/y_1]\dots[t_n/y_n]) = 1$

iff for all $G \in E(F)$, $\mathcal{A}(G) = 1$

iff $\mathcal{A}$ is a model of $E(F)$

Example

Let $F = \forall x\,(P(x) \vee Q(f(x))$.

Herbrand universe:

$$T(F) = \{f^k(a) \mid k \geq 0\} = \{a, f(a), f(f(a), \cdots\}$$

Herbrand expansion:

$$\begin{aligned} E(F) &= \{P(f^k(a)) \vee Q(f^{k+1}(a)) \mid k \geq 0\} \\ &= \{P(a) \vee Q(f(a)), P(f(a)) \vee Q(f^2(a)), P(f^2(a)) \vee Q(f^3(a)), \cdots\} \end{aligned}$$

$\mathcal{A}$ is a Herbrand model of F

iff for all $k \geq 0$, $\mathcal{A}[f^k(a)/x](P(x) \vee Q(f(x))) = 1$

iff for all $k \geq 0$, $\mathcal{A}(P(x) \vee Q(f(x))[f^k(a)/x]) = 1$

iff for all $k \geq 0$, $\mathcal{A}(P(f^k(a)) \vee Q(f^{k+1}(a)) = 1$

iff $\mathcal{A}$ is a model of $E(F)$

Herbrand's Theorem

Theorem

A closed formula F in Skolem form is unsatisfiable iff some finite subset of $E(F)$ is unsatisfiable.

Proof. Follows immediately from the Gödel-Herbrand-Skolem Theorem and the Compactness Theorem.

Example

We show that

$$F = \exists x \, \forall y \, P(x, y) \rightarrow \forall y \, \exists x \, P(x, y)$$

is valid, or, equivalently, that

$$\neg F \equiv \exists x \, \forall y \, P(x, y) \wedge \exists y \, \forall x \, \neg P(x, y)$$

is unsatisfiable.

Rectified form: $\exists x \, \forall y \, P(x, y) \wedge \exists z \, \forall v \, \neg P(v, z)$

Prenex form: $\exists x \, \exists z \, \forall y \, \forall v \, (P(x, y) \wedge \neg P(v, z))$

Skolem form: $\forall y \, \forall v \, (P(a, y) \wedge \neg P(v, b))$

Herbrand universe: $\{a, b\}$

Herbrand expansion: $\{ \; P(a, a) \wedge \neg P(a, b) \; , \; P(a, a) \wedge \neg P(b, b) \; ,$
$P(a, b) \wedge \neg P(a, b) \; , \; P(a, b) \wedge \neg P(b, b) \; \}$

Semi-decidability Theorems

Theorem

(a) *The unsatisfiability problem of predicate logic is (only) semi-decidable.*

(b) *The validity problem of predicate logic is (only) semi-decidable.*

Proof. (a) Gilmore's algorithm is a semi-decision procedure.

(The problem is undecidable. Proof later)

(b) F valid iff $\neg F$ unsatisfiable.

Gilmore's Algorithm

Let F be a closed formula in Skolem form
and let $F_1, F_2, F_3, \ldots$ be a computable enumeration of $E(F)$.

Input: F

$n := 0$;
repeat $n := n + 1$;
until $(F_1 \wedge F_2 \wedge \ldots \wedge F_n)$ is unsatisfiable;
return "unsatisfiable"

The algorithm terminates iff F is unsatisfiable.

Löwenheim-Skolem Theorem

Theorem

Every satisfiable formula of first-order predicate logic has a model with a countable universe.

Proof Let F_0 be a formula with free variables $x_1, \ldots, x_n$ for $n \geq 0$. Define $F := \exists x_1 \ldots \exists x_n\, F_0$ and observe that F_0 has a model with universe U iff F has a model with universe U.

Let G be closed formula in Skolem form equisatisfiable with F as produced by the Normal Form transformations starting with F.
Fact: Every model of G is a model of F.

$$
\begin{array}{lll}
F_0 \text{ satisfiable} & \Rightarrow & F \text{ satisfiable} \\
 & \Rightarrow & G \text{ satisfiable} \\
 & \Rightarrow & G \text{ has a Herbrand model} \\
 & \Rightarrow & F \text{ has a model with universe } T(G) \\
 & \Rightarrow & F_0 \text{ has a model with universe } T(G) \\
 & \Rightarrow & F_0 \text{ has a model with countable universe} \\
 & & (T(G) \text{ is countable})
\end{array}
$$

Löwenheim-Skolem Theorem

Formulas of first-order logic cannot enforce uncountable models