First-Order Logic
Normal Forms

Abbreviations

We return to the abbreviations used in connection with resolution:

$F_1 \to F_2$ abbreviates $\neg F_1 \vee F_2$

$\top$ abbreviates $P_1^0 \vee \neg P_1^0$

$\bot$ abbreviates $P_1^0 \wedge \neg P_1^0$

Substitution

- Substitutions are mappings from variables to terms.
- By $[t/x]$ we denote the substitution that replaces x by t.
- The notation $F[t/x]$ ("F with t for x") denotes the result of replacing all FREE *free* occurrences of x in F by t.

  Example: $(\forall x\ P(x) \wedge Q(x))[f(y)/x] = \forall x\ P(x) \wedge Q(f(y))$
- Similarly for subsitutions in terms:
  $u[t/x]$ is the result of replacing x by t in term u.

  Example: $(f(x))[g(x)/x] = f(g(x))$

Variable capture

If a term t of F contains a bound occurrence of a variable, substitution may lead to variable capture:

$$(\forall x\ P(x, y))[f(x)/y] \ = \ \forall x\ P(x, f(x))$$

Variable capture must be avoided

Substitution lemmas

Lemma

$\mathcal{A}(u[t/x]) \;=\; (\mathcal{A}[\mathcal{A}(t)/x])(u).$

Proof by structural induction on u.

Lemma (Substitution Lemma)

If t contains no variable bound in F then

$$\mathcal{A}(F[t/x]) \;=\; (\mathcal{A}[\mathcal{A}(t)/x])(F).$$

Proof by structural induction on F with the help of the lemma on terms.

Warning

The notation $.[./.]$ is heavily overloaded:

Substitution in syntactic objects

$F[G/A]$ in propositional logic

$F[t/x]$

$u[t/x]$ where u is a term

Function update

$\mathcal{A}[v/A]$ where $\mathcal{A}$ is a propositional assignment

$\mathcal{A}[d/x]$ where $\mathcal{A}$ is a structure and $d \in U_{\mathcal{A}}$

Overall goal

Transform any formula $F$of length m into a closed formula

$$\forall x_1 \ldots \forall x_n\, G \quad \text{where } G \text{ is quantifier-free,}$$

of lengt $O(m)$ that is equisatisfiable with F.

Rectified Formulas

Definition

A formula is rectified if no variable occurs both bound and free and all quantifiers in the formula bind different variables.

Lemma

Let $F = Qx\, G$ be a formula where $Q \in \{\forall, \exists\}$.
Let y be a variable that does not occur in G.
Then $F \equiv Qy\, G[y/x]$.

Lemma

Every formula is equivalent to a rectified formula.

Example

$$\forall x\, P(x, y) \land \exists x \exists y\, Q(x, y) \;\equiv\; \forall x'\, P(x', y) \land \exists x \exists y'\, Q(x, y')$$

Prenex form

Definition

A formula is in prenex form if it has the form

$$Q_1 y_1 \dots Q_n y_n \; F$$

where $Q_i \in \{\exists, \forall\}$, $n \geq 0$, and F is quantifier-free.

Prenex form

Theorem

Every formula is equivalent to a rectified formula in prenex form (a formula in **RPF***).*

Proof First construct an equivalent rectified formula.
Then pull the quantifiers to the front using the following equivalences from left to right as long as possible:

$$\begin{aligned}
\neg\forall x\, F &\equiv \exists x\, \neg F\\
\neg\exists x\, F &\equiv \forall x\, \neg F\\
Qx\, F \wedge G &\equiv Qx\, (F \wedge G)\\
F \wedge Qx\, G &\equiv Qx\, (F \wedge G)\\
Qx\, F \vee G &\equiv Qx\, (F \vee G)\\
F \vee Qx\, G &\equiv Qx\, (F \vee G)
\end{aligned}$$

For the last four rules note that the formula is rectified!

Skolem form

The Skolem form of a formula F in RPF is the result of applying the following algorithm to F:

while F contains an existential quantifier **do**

Let $F = \forall y_1 \forall y_2 \ldots \forall y_n \exists z\ G$
(the block of universal quantifiers may be empty)

Let f be a fresh function symbol of arity n
that does not occur in F.

$F := \forall y_1 \forall y_2 \ldots \forall y_n\ G[f(y_1, y_2, \ldots, y_n)/z]$

i.e. remove the outermost existential quantifier in F and replace every occurrence of z in G by $f(y_1, y_2, \ldots, y_n)$

Example

$\exists x\, \forall y\, \exists z\, \forall u\, \exists v\, P(x, y, z, u, v) \equiv$

Exercise

Which formulas are rectified, in prenex, or Skolem form?

	R	P	S
$\forall x\, (T(x) \vee C(x) \vee D(x))$			
$\exists x\, \exists y\, (C(y) \vee B(x, y))$			
$\neg \exists x\, C(x) \leftrightarrow \forall x\, \neg C(x)$			
$\forall x\, (C(x) \rightarrow S(x)) \rightarrow \forall y\, (\neg C(y) \rightarrow \neg S(y))$			

Skolem form

Theorem

A formula in RPF and its Skolem form are equisatisfiable.

Proof Show: Every iteration produces an equisatisfiable formula.
Let (for simplicity) $F = \forall y \, \exists z \, G$ and $F' = \forall y \, G[f(y)/z]$.

1. $F' \models F$, that is, every model of F' is a model of F.

Assume $\mathcal{A}$ is suitable for F' and $\mathcal{A}(F') = 1$.

$\Rightarrow$ for all $u \in U_{\mathcal{A}}$, $\mathcal{A}[u/y](G[f(y)/z]) = 1$

$\Rightarrow$ for all $u \in U_{\mathcal{A}}$, $\mathcal{A}[u/y][f^{\mathcal{A}}(u)/z](G) = 1$

$\Rightarrow$ for all $u \in U_{\mathcal{A}}$ there is a $v \in U_{\mathcal{A}}$ s.t. $\mathcal{A}[u/y][v/z](G) = 1$

$\Rightarrow$ for all $u \in U_{\mathcal{A}}$, $\mathcal{A}(\exists z \, G) = 1$

$\Rightarrow$ $\mathcal{A}(\forall y \, \exists z \, G) = 1$

Skolem form

Theorem

A formula in RPF and its Skolem form are equisatisfiable.

Proof Show: Every iteration produces an equisatisfiable formula.
Let (for simplicity) $F = \forall y\, \exists z\, G$ and $F' = \forall y\, G[f(y)/z]$.

2. If F has a model, so does F'

Assume $\mathcal{A}$ is suitable for F and $\mathcal{A}(F) = 1$.

W.l.o.g. $\mathcal{A}$ does not define f (because f is new).

$\Rightarrow$ for all $u \in U_{\mathcal{A}}$ there is $v \in U_{\mathcal{A}}$ s.t. $\mathcal{A}[u/y][v/z](G) = 1 \quad (*)$

Let $\mathcal{A}'$ be $\mathcal{A}$ extended with a definition of f: $f^{\mathcal{A}'}(u) := v$, where v is chosen as in $(*)$.

$\Rightarrow \mathcal{A}'(F') = 1$ because for all $u \in U_{\mathcal{A}}$:

$$
\begin{aligned}
\mathcal{A}'[u/y](G[f(y)/z]) &= \mathcal{A}'[u/y][f^{\mathcal{A}'[u/y]}(u)/z](G) && \text{(subs. lemma)}\\
&= \mathcal{A}'[u/y][f^{\mathcal{A}'}(u)/z](G) && \text{(def. of } \mathcal{A}')\\
&= \mathcal{A}'[u/y][v/z](G) = 1 && \text{(def. of } \mathcal{A}' \text{ and (*))}
\end{aligned}
$$

Summary: conversion to Skolem form

Input: a formula F

Output: a rectified, closed formula in Skolem form $\forall y_1 \dots \forall y_k \; G$, where G is quantifier-free, that is equisatisfiable with F.

1. Rectify F by systematic renaming of bound variables.
   The result is a formula F_1 equivalent to F.
2. Let $y_1, y_2, \dots, y_n$ be the variables occurring free in F_1.
   Produce the formula $F_2 = \exists y_1 \exists y_2 \dots \exists y_n \, F_1$.
   F_2 is equisatisfiable with F_1, rectified and closed.
3. Produce a formula F_3 in RPF equivalent to F_2.
4. Eliminate the existential quantifiers in F_3
   by transforming F_3 into its Skolem form F_4.
   The formula F_4 is equisatisfiable with F_3.

Exercise

Convert into Skolem form $F = \forall x\, P(y, f(x, y)) \vee \neg\forall y\, Q(g(x), y)$