

Hilbert Systems

Propositional Logic

(See the book by Troelstra and Schwichtenberg)

Easy to define, hard to use.
No context management.

Hilbert systems and proof trees

A Hilbert system for propositional logic consists of

- ▶ a set of axioms (formulas over formula variables)
- ▶ and a single inference rule, $\rightarrow E$ or modus ponens:

$$\frac{F \rightarrow G \quad F}{G} \rightarrow E$$

Nodes of a proof tree are (labeled with) formulas.

Proof trees are defined inductively:

- ▶ Every formula F is a proof tree.
- ▶ Larger proof trees are constructed using $\rightarrow E$ (and $\rightarrow E$ only).

$\Gamma \vdash_H F$ denotes that there is a proof tree with root F whose leaves are either instances of axioms or elements of Γ (assumptions).

Alternative presentation

Proofs in Hilbert systems are frequently shown as lists of lines

$$\begin{array}{l} 1: F_1 \quad \textit{justification}_1 \\ 2: F_2 \quad \textit{justification}_2 \\ \vdots \\ i: F_i \quad \textit{justification}_i \\ \vdots \\ n: F_n \quad \textit{justification}_n \end{array}$$

where $\textit{justification}_i$ is either $\left\{ \begin{array}{l} \textit{assumption}, \\ \textit{axiom}, \\ \rightarrow E (j, k), \text{ with } j, k < i. \end{array} \right.$

Notational convention:

$$F \rightarrow G \rightarrow H \text{ means } F \rightarrow (G \rightarrow H)$$

Note: $F \rightarrow G \rightarrow H \equiv F \wedge G \rightarrow H$
 $F \rightarrow G \rightarrow H \not\equiv (F \rightarrow G) \rightarrow H$

A simple Hilbert system

Axioms: $F \rightarrow G \rightarrow F$ (A1)

$(F \rightarrow G \rightarrow H) \rightarrow (F \rightarrow G) \rightarrow F \rightarrow H$ (A2)

A proof of $A \rightarrow A$:

A simple Hilbert system

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A proof of $A \rightarrow A$:

1 :

2 :

3 :

4 :

5 : $A \rightarrow A$

A simple Hilbert system

Axioms: $F \rightarrow G \rightarrow F$ (A1)

$(F \rightarrow G \rightarrow H) \rightarrow (F \rightarrow G) \rightarrow F \rightarrow H$ (A2)

A proof of $A \rightarrow A$:

1 :

2 :

3 : _____ $\rightarrow A \rightarrow A$

4 : _____

5 : $A \rightarrow A$ $\rightarrow E : 3, 4$

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Axioms: $F \rightarrow G \rightarrow F$ (A1)

$(F \rightarrow G \rightarrow H) \rightarrow (F \rightarrow G) \rightarrow F \rightarrow H$ (A2)

A proof of $A \rightarrow A$:

1 :

2 :

3 : $(A \rightarrow A \rightarrow A) \rightarrow A \rightarrow A$

4 : $A \rightarrow A \rightarrow A$

5 : $A \rightarrow A$

$\rightarrow E : 3, 4$

A simple Hilbert system

Axioms: $F \rightarrow G \rightarrow F$ (A1)

$(F \rightarrow G \rightarrow H) \rightarrow (F \rightarrow G) \rightarrow F \rightarrow H$ (A2)

A proof of $A \rightarrow A$:

1 :

2 :

3 : $(A \rightarrow A \rightarrow A) \rightarrow A \rightarrow A$

4 : $A \rightarrow A \rightarrow A$

A1

5 : $A \rightarrow A$

$\rightarrow E$: 3, 4

A simple Hilbert system

Axioms: $F \rightarrow G \rightarrow F$ (A1)

$(F \rightarrow G \rightarrow H) \rightarrow (F \rightarrow G) \rightarrow F \rightarrow H$ (A2)

A proof of $A \rightarrow A$:

1: _____ $\rightarrow (A \rightarrow A \rightarrow A) \rightarrow A \rightarrow A$

2: _____

3: $(A \rightarrow A \rightarrow A) \rightarrow A \rightarrow A$ $\rightarrow E : 2, 1$

4: $A \rightarrow A \rightarrow A$ A1

5: $A \rightarrow A$ $\rightarrow E : 3, 4$

A simple Hilbert system

Axioms: $F \rightarrow G \rightarrow F$ (A1)

$(F \rightarrow G \rightarrow H) \rightarrow (F \rightarrow G) \rightarrow F \rightarrow H$ (A2)

A proof of $A \rightarrow A$:

1 : $(A \rightarrow (A \rightarrow A) \rightarrow A) \rightarrow (A \rightarrow A \rightarrow A) \rightarrow A \rightarrow A$

2 : $A \rightarrow (A \rightarrow A) \rightarrow A$

3 : $(A \rightarrow A \rightarrow A) \rightarrow A \rightarrow A$ $\rightarrow E : 2, 1$

4 : $A \rightarrow A \rightarrow A$ A1

5 : $A \rightarrow A$ $\rightarrow E : 3, 4$

A simple Hilbert system

Axioms: $F \rightarrow G \rightarrow F$ (A1)

$(F \rightarrow G \rightarrow H) \rightarrow (F \rightarrow G) \rightarrow F \rightarrow H$ (A2)

A proof of $A \rightarrow A$:

1 : $(A \rightarrow (A \rightarrow A) \rightarrow A) \rightarrow (A \rightarrow A \rightarrow A) \rightarrow A \rightarrow A$

2 : $A \rightarrow (A \rightarrow A) \rightarrow A$ A1

3 : $(A \rightarrow A \rightarrow A) \rightarrow A \rightarrow A$ $\rightarrow E : 2, 1$

4 : $A \rightarrow A \rightarrow A$ A1

5 : $A \rightarrow A$ $\rightarrow E : 3, 4$

A simple Hilbert system

Axioms: $F \rightarrow G \rightarrow F$ (A1)

$(F \rightarrow G \rightarrow H) \rightarrow (F \rightarrow G) \rightarrow F \rightarrow H$ (A2)

A proof of $A \rightarrow A$:

1 : $(A \rightarrow (A \rightarrow A) \rightarrow A) \rightarrow (A \rightarrow A \rightarrow A) \rightarrow A \rightarrow A$ A2

2 : $A \rightarrow (A \rightarrow A) \rightarrow A$ A1

3 : $(A \rightarrow A \rightarrow A) \rightarrow A \rightarrow A$ $\rightarrow E : 2, 1$

4 : $A \rightarrow A \rightarrow A$ A1

5 : $A \rightarrow A$ $\rightarrow E : 3, 4$

A simple Hilbert system

Axioms: $F \rightarrow G \rightarrow F$ (A1)

$(F \rightarrow G \rightarrow H) \rightarrow (F \rightarrow G) \rightarrow F \rightarrow H$ (A2)

A proof of $A \rightarrow A$:

1 : $(A \rightarrow (A \rightarrow A) \rightarrow A) \rightarrow (A \rightarrow A \rightarrow A) \rightarrow A \rightarrow A$ A2

2 : $A \rightarrow (A \rightarrow A) \rightarrow A$ A1

3 : $(A \rightarrow A \rightarrow A) \rightarrow A \rightarrow A$ $\rightarrow E : 2, 1$

4 : $A \rightarrow A \rightarrow A$ A1

5 : $A \rightarrow A$ $\rightarrow E : 3, 4$

$\Rightarrow \vdash_H A \rightarrow A$

A simple Hilbert system

Axioms: $F \rightarrow G \rightarrow F$ (A1)

$(F \rightarrow G \rightarrow H) \rightarrow (F \rightarrow G) \rightarrow F \rightarrow H$ (A2)

A proof of $A \rightarrow A$:

1 : $(A \rightarrow (A \rightarrow A) \rightarrow A) \rightarrow (A \rightarrow A \rightarrow A) \rightarrow A \rightarrow A$ A2

2 : $A \rightarrow (A \rightarrow A) \rightarrow A$ A1

3 : $(A \rightarrow A \rightarrow A) \rightarrow A \rightarrow A$ $\rightarrow E : 2, 1$

4 : $A \rightarrow A \rightarrow A$ A1

5 : $A \rightarrow A$ $\rightarrow E : 3, 4$

$\Rightarrow \vdash_H A \rightarrow A$

Observe: The same proof can be used to derive $F \rightarrow F$ for any formula F .

Theorem (Deduction Theorem)

In any Hilbert-system that contains the axioms A1 and A2:

$$F, \Gamma \vdash_H G \quad \text{iff} \quad \Gamma \vdash_H F \rightarrow G$$

Proof “ \Leftarrow ”: Assume $\Gamma \vdash_H F \rightarrow G$. We prove $F, \Gamma \vdash_H G$.

$$\begin{aligned} & \Gamma \vdash_H F \rightarrow G \\ \Rightarrow & F, \Gamma \vdash_H F \rightarrow G \\ \Rightarrow & F, \Gamma \vdash_H G \quad \text{by } \rightarrow E \text{ because } F, \Gamma \vdash_H F \end{aligned}$$

Theorem (Deduction Theorem)

In any Hilbert-system that contains the axioms A1 and A2:

$$F, \Gamma \vdash_H G \quad \text{iff} \quad \Gamma \vdash_H F \rightarrow G$$

Proof “ \Rightarrow ”: Assume $F, \Gamma \vdash_H G$ with a proof of length n .
We prove $\Gamma \vdash_H F \rightarrow G$ by induction on n .

Base: $n = 1$. Then either $G \in \Gamma \cup \{F\}$ or G is instance of axiom.

- ▶ $G = F$. To prove: $\Gamma \vdash_H F \rightarrow F$. Done earlier.
- ▶ $G \in \Gamma$ or instance of axiom.

1 : G

2 : $G \rightarrow F \rightarrow G$

A1

3 : $F \rightarrow G$

$\rightarrow E : 1, 2$

Theorem (Deduction Theorem)

In any Hilbert-system that contains the axioms A1 and A2:

$$F, \Gamma \vdash_H G \quad \text{iff} \quad \Gamma \vdash_H F \rightarrow G$$

Proof “ \Rightarrow ”: Assume $F, \Gamma \vdash_H G$ with a proof of length n .
We prove $\Gamma \vdash_H F \rightarrow G$ by induction on n .

Step: $n > 1$. Assume last $\rightarrow E$ gives G from $H \rightarrow G$ and H .

IH: $\Gamma \vdash_H F \rightarrow H$ and $\Gamma \vdash_H F \rightarrow H \rightarrow G$.

To prove: $\Gamma \vdash_H F \rightarrow G$.

$$\begin{array}{c} \text{A2: } (F \rightarrow H \rightarrow G) \rightarrow (F \rightarrow H) \rightarrow F \rightarrow G \quad F \rightarrow H \rightarrow G \\ \hline (F \rightarrow H) \rightarrow F \rightarrow G \quad F \rightarrow H \\ \hline F \rightarrow G \end{array}$$

Hilbert System

From now on \vdash_H refers to the following set of axioms:

$$F \rightarrow G \rightarrow F \quad (\text{A1})$$

$$(F \rightarrow G \rightarrow H) \rightarrow (F \rightarrow G) \rightarrow F \rightarrow H \quad (\text{A2})$$

$$F \rightarrow G \rightarrow F \wedge G \quad (\text{A3})$$

$$F \wedge G \rightarrow F \quad (\text{A4})$$

$$F \wedge G \rightarrow G \quad (\text{A5})$$

$$F \rightarrow F \vee G \quad (\text{A6})$$

$$G \rightarrow F \vee G \quad (\text{A7})$$

$$F \vee G \rightarrow (F \rightarrow H) \rightarrow (G \rightarrow H) \rightarrow H \quad (\text{A8})$$

$$(\neg F \rightarrow \perp) \rightarrow F \quad (\text{A9})$$

We prove soundness and completeness.

Relating Hilbert and Natural Deduction

Theorem (Hilbert can simulate ND)

If $\Gamma \vdash_N F$ then $\Gamma \vdash_H F$

Proof. Translation in two steps:

$$\vdash_N \overset{(1)}{\rightsquigarrow} \vdash_H + \rightarrow I \overset{(2)}{\rightsquigarrow} \vdash_H$$

1. Transform a ND-proof tree into a proof tree containing Hilbert axioms, $\rightarrow E$, and $\rightarrow I$ by replacing all other ND rules by Hilbert proofs with $\rightarrow I$

Principle: ND rule \rightsquigarrow 1 axiom + $\rightarrow I/E$

2. Eliminate $\rightarrow I$ rules using the Deduction Theorem

Theorem (ND can simulate Hilbert)

If $\Gamma \vdash_H F$ then $\Gamma \vdash_N F$

Proof by induction on the length of the Hilbert proof of F .

- ▶ Every Hilbert axiom is provable in ND (Exercise!).
- ▶ $\rightarrow E$ is also available in ND.

Corollary

$\Gamma \vdash_H F$ iff $\Gamma \vdash_N F$

Corollary (Soundness and completeness)

$\Gamma \vdash_H F$ iff $\Gamma \models F$