Natural Deduction
Propositional Logic

(See the book by Troelstra and Schwichtenberg)

Natural deduction (Gentzen 1935) aims at *natural* proofs.

It formalizes good mathematical practice.

Resolution, but also sequent calculus, aim at proof search.

Main principles: Introduction and elimination rules

1. For every logical operator $\oplus$ there are two kinds of rules:

- Introduction rules: How to prove $F \oplus G$

$$\frac{\ldots}{F \oplus G}$$

- Elimination rules: What can be proved from $F \oplus G$

$$\frac{F \oplus G \quad \ldots}{\ldots}$$

Examples

$$\frac{A \quad B}{A \land B} \;\land I \qquad \frac{F \land G}{F} \;\land E_1 \qquad \frac{F \land G}{G} \;\land E_2$$

Main principles: Local assumptions

2. Proof can contain subproofs with *local/closed* assumptions

Example

Inference rule formalizing "if from the local assumption F we can prove G then we can prove $F \rightarrow G$":

$$\dfrac{\begin{array}{c}[F] \\ \vdots \\ G\end{array}}{F \rightarrow G} \; \rightarrow I$$

A proof tree:

$$\dfrac{\dfrac{[P] \quad Q}{P \wedge Q} \; \wedge I}{P \rightarrow P \wedge Q} \; \rightarrow I$$

"From the (open) assumption Q we can prove $P \rightarrow P \wedge Q$."
In symbols: $Q \vdash_N P \rightarrow P \wedge Q$

Main principles: Growing the proof tree

Upwards:

Main principles: Growing the proof tree

Upwards:

$$\overline{P \to P \land Q}$$

Main principles: Growing the proof tree

Upwards:

$$\dfrac{}{P \rightarrow P \wedge Q} \quad {\rightarrow}I$$

Main principles: Growing the proof tree

Upwards:

$$\dfrac{\dfrac{}{P \wedge Q}}{P \to P \wedge Q} \; {\to}I$$

Main principles: Growing the proof tree

Upwards:

$$\dfrac{\dfrac{}{P \wedge Q}\ \wedge I}{P \rightarrow P \wedge Q}\ \rightarrow I$$

Main principles: Growing the proof tree

Upwards:

$$\dfrac{\dfrac{P \qquad Q}{P \wedge Q}\ \wedge I}{P \rightarrow P \wedge Q}\ \rightarrow I$$

Main principles: Growing the proof tree

Upwards:

$$\dfrac{\dfrac{[P] \quad Q}{P \wedge Q}\ \wedge I}{P \to P \wedge Q}\ \to I$$

Main principles: Growing the proof tree

Upwards:

$$\dfrac{\dfrac{[P] \quad Q}{P \wedge Q}\ \wedge I}{P \to P \wedge Q}\ \to I$$

Downwards:

Main principles: Growing the proof tree

Upwards:

$$\cfrac{\cfrac{[P] \quad Q}{P \wedge Q}\ \wedge I}{P \rightarrow P \wedge Q}\ \rightarrow I$$

Downwards:

Main principles: Growing the proof tree

Upwards:

$$\dfrac{\dfrac{[P] \quad Q}{P \land Q}\ \land I}{P \to P \land Q}\ \to I$$

Downwards:

$$\dfrac{P \quad Q}{\qquad\qquad}$$

Main principles: Growing the proof tree

Upwards:

$$\dfrac{\dfrac{[P] \quad Q}{P \wedge Q}\ \wedge I}{P \rightarrow P \wedge Q}\ \rightarrow I$$

Downwards:

$$\dfrac{P \quad Q}{}\ \wedge I$$

Main principles: Growing the proof tree

Upwards:

$$\dfrac{\dfrac{[P] \quad Q}{P \wedge Q}\ \wedge I}{P \rightarrow P \wedge Q}\ \rightarrow I$$

Downwards:

$$\dfrac{\dfrac{P \quad Q}{P \wedge Q}\ \wedge I}{\quad}$$

Main principles: Growing the proof tree

Upwards:

$$\dfrac{\dfrac{[P] \quad Q}{P \wedge Q}\ \wedge I}{P \rightarrow P \wedge Q}\ \rightarrow I$$

Downwards:

$$\dfrac{\dfrac{P \quad Q}{P \wedge Q}\ \wedge I}{}\ \rightarrow I$$

Main principles: Growing the proof tree

Upwards:

$$\dfrac{\dfrac{[P] \quad Q}{P \wedge Q} \; \wedge I}{P \rightarrow P \wedge Q} \; \rightarrow I$$

Downwards:

$$\dfrac{\dfrac{P \quad Q}{P \wedge Q} \; \wedge I}{P \rightarrow P \wedge Q} \; \rightarrow I$$

Main principles: Growing the proof tree

Upwards:

$$\dfrac{\dfrac{[P] \quad Q}{P \wedge Q} \wedge I}{P \rightarrow P \wedge Q} \rightarrow I$$

Downwards:

$$\dfrac{\dfrac{[P] \quad Q}{P \wedge Q} \wedge I}{P \rightarrow P \wedge Q} \rightarrow I$$

ND proof trees

- The nodes of a ND proof tree are (labeled by) formulas.
- Leaf nodes are called assumptions.
- The root is called the conclusion.

- Assumptions can be open or closed.
- Closed assumptions are written [F].
- $\Gamma \vdash_N F$ denotes that there is a proof tree with conclusion F whose open assumptions belong to the set of formulas Γ. (Reading: F is provable (derivable) from Γ.)

Intuition:

- A proof tree shows that the conjunction of the open assumptions entails the conclusion.

- Closed assumptions are auxiliary local assumptions in a subproof that have been closed ("discharged") by some proof rule like $\rightarrow I$.

ND proof trees

ND proof trees are defined inductively:

- Every formula F is a ND proof tree with open assumption F and conclusion F.
  (Intuition: From F we can prove F.)
- Larger proof trees are constructed using the rules of ND:
  - Introduction and Elimination rules for $\wedge, \vee, \rightarrow, \neg$, plus
  - a rule for $\bot$.

  The application of a rule (backwards or forwards) adds new nodes to the tree and possibly closes some assumptions:

Natural Deduction rules

$$\frac{F \quad G}{F \wedge G}\ \wedge I \qquad \frac{F \wedge G}{F}\ \wedge E_1 \quad \frac{F \wedge G}{G}\ \wedge E_2$$

$$\dfrac{\begin{array}{c}[F]\\ \vdots \\ G\end{array}}{F \rightarrow G}\ \rightarrow I \qquad \frac{F \rightarrow G \quad F}{G}\ \rightarrow E$$

$$\frac{F}{F \vee G}\ \vee I_1 \quad \frac{G}{F \vee G}\ \vee I_2 \qquad \dfrac{F \vee G \quad \begin{array}{c}[F]\\ \vdots \\ H\end{array} \quad \begin{array}{c}[G]\\ \vdots \\ H\end{array}}{H}\ \vee E$$

$$\dfrac{\begin{array}{c}[F]\\ \vdots \\ \bot\end{array}}{\neg F}\ \neg I \qquad \frac{\neg F \quad F}{\bot}\ \neg E \qquad \dfrac{\begin{array}{c}[\neg F]\\ \vdots \\ \bot\end{array}}{F}\ \bot$$

Natural Deduction rules

How to read a rule

$$\dfrac{\dots \quad \begin{array}{c}[F] \\ \vdots \\ G\end{array} \quad \dots}{\dots}\ r$$

Forward:
When applying rule r, in the proof of G we can close all (or some) of the assumptions F.

Backward:
In the subproof of G we can use the local assumption $[F]$.

We can use labels to show which rule application closed which assumptions (the slides won't but you must!).

Examples of proofs

$P \to Q \vdash_N \neg Q \to \neg P$:

$$\dfrac{\dfrac{\dfrac{\dfrac{[P]^2 \quad P \to Q}{Q}\;{\to}E{:}4 \qquad [\neg Q]^1}{\bot}\;\neg E{:}3}{\neg P}\;\neg I{:}2}{\neg Q \to \neg P}\;{\to}I{:}1$$

$\neg(P \vee Q) \vdash_N \neg P \wedge \neg Q$:

$$\dfrac{\dfrac{\dfrac{\dfrac{[P]^2}{P \vee Q}\;\vee I{:}4 \qquad \neg(P \vee Q)}{\bot}\;\neg E{:}3}{\neg P}\;\neg I{:}2 \qquad \dfrac{\dfrac{\dfrac{[Q]^5}{P \vee Q}\;\vee I{:}7 \qquad \neg(P \vee Q)}{\bot}\;\neg E{:}6}{\neg Q}\;\neg I{:}5}{\neg P \wedge \neg Q}\;\vee I{:}1$$

Soundness

Lemma (Soundness)

If $\Gamma \vdash_N F$ *then* $\Gamma \models F$

Proof by induction on the depth of the proof tree for $\Gamma \vdash_N F$.

Base: The tree has only one node F and $F \in \Gamma$.

Step: Case analysis of first rule applied (upwards).

Case: first rule is $\dfrac{G \rightarrow F \quad G}{F} \rightarrow E$

Let $\mathcal{A}$ arbitrary such that $\mathcal{A}(\Gamma) = 1$. We prove $\mathcal{A}(F) = 1$

IH: $\Gamma \models G \rightarrow F$ and $\Gamma \models G$

IH and $\mathcal{A}(\Gamma) = 1$ yields $\mathcal{A}(G \rightarrow F) = 1$ and $\mathcal{A}(G) = 1$.
So $\mathcal{A}(F) = 1$

Soundness

Case: first rule is
$$\dfrac{\begin{matrix}[G]\\ \vdots \\ F\end{matrix}}{G \to F} \to I$$

To show: $\Gamma \models G \to F$

IH: $\Gamma, G \models F$

$$\begin{array}{ll} & \Gamma \models G \to F \\ \text{iff} & \text{for all } \mathcal{A}\text{:} \;\; \mathcal{A} \models \Gamma \Rightarrow \mathcal{A} \models G \to F \\ \text{iff} & \text{for all } \mathcal{A}\text{:} \;\; \mathcal{A} \models \Gamma \Rightarrow (\mathcal{A} \models G \Rightarrow \mathcal{A} \models F) \\ \text{iff} & \text{for all } \mathcal{A}\text{:} \;\; (\mathcal{A} \models \Gamma \text{ and } \mathcal{A} \models G) \;\Rightarrow\; \mathcal{A} \models F \\ \text{iff} & \text{IH} \end{array}$$

Towards completeness: ND can simulate truth tables

Lemma (Tertium non datur)

$\vdash_N F \lor \neg F$

Proof:

$$\dfrac{\dfrac{[\neg(F \lor \neg F)]^1 \qquad \dfrac{\dfrac{\dfrac{[\neg(F \lor \neg F)]^1 \qquad \dfrac{[\neg F]^4}{F \lor \neg F}\ {\lor I_2{:}6}}{\bot}\ {\neg E{:}5}}{F}\ {\bot{:}4}}{F \lor \neg F}\ {\lor I_1{:}3}}{\bot}\ {\neg E{:}2}}{F \lor \neg F}\ {\bot{:}1}$$

Towards completeness: ND can simulate truth tables

Definition

$$F^{\mathcal{A}} := \begin{cases} F & \text{if } \mathcal{A}(F) = 1 \\ \neg F & \text{if } \mathcal{A}(F) = 0 \end{cases}$$

Lemma (1)

If atoms$(F) \subseteq \{A_1, \ldots, A_n\}$ *then* $A_1^{\mathcal{A}}, \ldots, A_n^{\mathcal{A}} \vdash_N F^{\mathcal{A}}$ *for every* $\mathcal{A}$*.*

Proof By induction on F.

Only the case $F = G \to H$. Three subcases:

$\mathcal{A}(H) = 1$. Then $H^{\mathcal{A}} = H$, $(G \to H)^{\mathcal{A}} = G \to H$.
To prove: $A_1^{\mathcal{A}}, \ldots, A_n^{\mathcal{A}} \vdash_N G \to H$. By IH: $A_1^{\mathcal{A}}, \ldots, A_n^{\mathcal{A}} \vdash_N H$.

$$\dfrac{\dfrac{}{H}\,IH}{G \to H}\,{\to}I$$

Towards completeness: ND can simulate truth tables

$\mathcal{A}(G) = 0$. Then $G^{\mathcal{A}} = \neg G$, $(G \to H)^{\mathcal{A}} = G \to H$.
To prove: $A_1^{\mathcal{A}}, \ldots, A_n^{\mathcal{A}} \vdash_N G \to H$. By IH: $A_1^{\mathcal{A}}, \ldots, A_n^{\mathcal{A}} \vdash_N \neg G$.

$$\dfrac{\dfrac{\dfrac{[G] \quad \dfrac{}{\neg G}\,IH}{\bot}\,\neg E}{H}\,\bot}{G \to H}\,\to I$$

$\mathcal{A}(G) = 1$ and $\mathcal{A}(H) = 0$. Then $G^{\mathcal{A}} = G$, $H^{\mathcal{A}} = \neg H$,
$(G \to H)^{\mathcal{A}} = \neg(G \to H)$.
To prove: $A_1^{\mathcal{A}}, \ldots, A_n^{\mathcal{A}} \vdash_N \neg(G \to H)$.
By IH: $A_1^{\mathcal{A}}, \ldots, A_n^{\mathcal{A}} \vdash_N G$ and $A_1^{\mathcal{A}}, \ldots, A_n^{\mathcal{A}} \vdash_N \neg H$.

$$\dfrac{\dfrac{\dfrac{[G \to H] \quad \dfrac{}{G}\,IH}{H}\,\to E \quad \dfrac{}{\neg H}\,IH}{\bot}\,\bot}{\neg(G \to H)}\,\to I$$

Towards completeness: ND can simulate truth tables

Corollary (Cases)

If $F, \Gamma \vdash_N G$ *and* $\neg F, \Gamma \vdash_N G$ *then* $\Gamma \vdash_N G$.

Proof: By Lemma (Tertium non datur) $F, \Gamma \vdash_N F \vee \neg F$.
Apply

$$\dfrac{F \vee \neg F \qquad \begin{matrix}[F] \\ \vdots \\ G\end{matrix} \qquad \begin{matrix}[\neg F] \\ \vdots \\ G\end{matrix}}{G} \; \vee E$$

Completeness

Lemma (2)

If $atoms(F) = \{A_1, \ldots, A_n\}$ and $\models F$ then $A_1^{\mathcal{A}}, \ldots, A_k^{\mathcal{A}} \vdash_N F$ for every $\mathcal{A}$ and for all $k \leq n$.

Proof by (downward) induction on $k = n, \ldots, 0$.

$k = n$. $A_1^{\mathcal{A}}, \ldots, A_n^{\mathcal{A}} \vdash_N F$ holds by Lemma (1) and $F^{\mathcal{A}} = F$ because F is valid.

$k < n$. By IH $A_1^{\mathcal{A}}, \ldots, A_k^{\mathcal{A}} \vdash_N F$ for every $\mathcal{A}$.
To prove: $A_1^{\mathcal{A}}, \ldots, A_{k-1}^{\mathcal{A}} \vdash_N F$ for every $\mathcal{A}$.
Let $\mathcal{A}$ arbitrary. Define $\overline{\mathcal{A}}$ by $\overline{\mathcal{A}}(A_i) = \mathcal{A}(A_i)$ for every $i < k$ and $\overline{\mathcal{A}}(A_k) = 1 - \mathcal{A}(A_k)$.
Assume w.l.o.g. $\mathcal{A}(A_k) = 1$ (otherwise swap $\mathcal{A}$ and $\overline{\mathcal{A}}$).
Then $A_i^{\overline{\mathcal{A}}} = A_i^{\mathcal{A}}$ for every $i < k$, and $A_k^{\overline{\mathcal{A}}} = \neg A_k^{\mathcal{A}}$.
Taking $F := \mathcal{A}_k^{\mathcal{A}}$, $\Gamma := A_1^{\mathcal{A}}, \ldots, A_{k-1}^{\mathcal{A}}$, and $G := F$ in Corollary (Cases) yields $A_1^{\mathcal{A}}, \ldots, A_{k-1}^{\mathcal{A}} \vdash_N F$.

Completeness

Theorem (Completeness)

If $\Gamma \models F$ *then* $\Gamma \vdash_N F$.

Proof Only for $\Gamma := G$ (general case left as exercise).

Assume $G \models F$.
We have $\models G \to F$ and so $\vdash_N G \to F$ by Lemma (2) with $n = 0$.
Applying

$$\frac{G \to F \quad G}{F} \to E$$

yields $G \vdash_N F$.