

Sequent Calculus

Propositional Logic

Sequent Calculus

Invented by Gerhard Gentzen in 1935. Birth of proof theory.

Proof rules

$$\frac{S_1 \quad \dots \quad S_n}{S}$$

where S_1, \dots, S_n and S are **sequents**: expressions of the form

$$\Gamma \Rightarrow \Delta$$

with Γ and Δ finite **multisets** of formulas.

Multiset = set with possibly repeated elements; using sets possible but less elegant.

Notice: \Rightarrow is just a—suggestive—separator

Intention of the calculus:

$$\Gamma \Rightarrow \Delta \text{ is provable (derivable) iff } \bigwedge \Gamma \models \bigvee \Delta \quad (\bigwedge \Gamma \rightarrow \bigvee \Delta \text{ valid})$$

Sequents: Notation

- ▶ We use set notation for multisets, e.g. $\{A, B \rightarrow C, A\}$
- ▶ Drop $\{\}$: $F_1, \dots, F_m \Rightarrow G_1, \dots, G_n$
- ▶ F, Γ abbreviates $\{F\} \cup \Gamma$ (similarly for Δ)
- ▶ Γ_1, Γ_2 abbreviates $\Gamma_1 \cup \Gamma_2$ (similarly for Δ)

Sequent Calculus rules

$$\frac{}{\perp, \Gamma \Rightarrow \Delta} \quad \perp L$$

$$\frac{\Gamma \Rightarrow F, \Delta}{\neg F, \Gamma \Rightarrow \Delta} \quad \neg L$$

$$\frac{F, G, \Gamma \Rightarrow \Delta}{F \wedge G, \Gamma \Rightarrow \Delta} \quad \wedge L$$

$$\frac{F, \Gamma \Rightarrow \Delta \quad G, \Gamma \Rightarrow \Delta}{F \vee G, \Gamma \Rightarrow \Delta} \quad \vee L$$

$$\frac{\Gamma \Rightarrow F, \Delta \quad G, \Gamma \Rightarrow \Delta}{F \rightarrow G, \Gamma \Rightarrow \Delta} \quad \rightarrow L$$

$$\frac{}{A, \Gamma \Rightarrow A, \Delta} \quad Ax$$

$$\frac{F, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \neg F, \Delta} \quad \neg R$$

$$\frac{\Gamma \Rightarrow F, \Delta \quad \Gamma \Rightarrow G, \Delta}{\Gamma \Rightarrow F \wedge G, \Delta} \quad \wedge R$$

$$\frac{\Gamma \Rightarrow F, G, \Delta}{\Gamma \Rightarrow F \vee G, \Delta} \quad \vee R$$

$$\frac{F, \Gamma \Rightarrow G, \Delta}{\Gamma \Rightarrow F \rightarrow G, \Delta} \quad \rightarrow R$$

Sequent Calculus rules

Intuition: read backwards as proof search rules

$$\frac{}{\perp, \Gamma \Rightarrow \Delta} \quad \perp L$$

$$\frac{\Gamma \Rightarrow F, \Delta}{\neg F, \Gamma \Rightarrow \Delta} \quad \neg L$$

$$\frac{F, G, \Gamma \Rightarrow \Delta}{F \wedge G, \Gamma \Rightarrow \Delta} \quad \wedge L$$

$$\frac{F, \Gamma \Rightarrow \Delta \quad G, \Gamma \Rightarrow \Delta}{F \vee G, \Gamma \Rightarrow \Delta} \quad \vee L$$

$$\frac{\Gamma \Rightarrow F, \Delta \quad G, \Gamma \Rightarrow \Delta}{F \rightarrow G, \Gamma \Rightarrow \Delta} \quad \rightarrow L$$

$$\frac{}{A, \Gamma \Rightarrow A, \Delta} \quad Ax$$

$$\frac{F, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \neg F, \Delta} \quad \neg R$$

$$\frac{\Gamma \Rightarrow F, \Delta \quad \Gamma \Rightarrow G, \Delta}{\Gamma \Rightarrow F \wedge G, \Delta} \quad \wedge R$$

$$\frac{\Gamma \Rightarrow F, G, \Delta}{\Gamma \Rightarrow F \vee G, \Delta} \quad \vee R$$

$$\frac{F, \Gamma \Rightarrow G, \Delta}{\Gamma \Rightarrow F \rightarrow G, \Delta} \quad \rightarrow R$$

Every rule decomposes its principal formula

$$\overline{\Rightarrow (P \vee R) \wedge (Q \vee \neg R) \rightarrow P \vee Q}$$

$$\frac{}{\Rightarrow (P \vee R) \wedge (Q \vee \neg R) \rightarrow P \vee Q} \rightarrow R$$

$$\frac{F, \Gamma \Rightarrow G, \Delta}{\Gamma \Rightarrow F \rightarrow G, \Delta} \rightarrow R$$

$$\frac{\overline{(P \vee R) \wedge (Q \vee \neg R) \Rightarrow P \vee Q}}{\Rightarrow (P \vee R) \wedge (Q \vee \neg R) \rightarrow P \vee Q} \rightarrow R$$

$$\frac{F, \Gamma \Rightarrow G, \Delta}{\Gamma \Rightarrow F \rightarrow G, \Delta} \rightarrow R$$

$$\frac{\overline{(P \vee R) \wedge (Q \vee \neg R) \Rightarrow P \vee Q}^{\wedge L}}{\Rightarrow (P \vee R) \wedge (Q \vee \neg R) \rightarrow P \vee Q} \rightarrow R$$

$$\frac{F, G, \Gamma \Rightarrow \Delta}{F \wedge G, \Gamma \Rightarrow \Delta} \wedge L$$

$$\frac{\overline{P \vee R, Q \vee \neg R \Rightarrow P \vee Q}}{(P \vee R) \wedge (Q \vee \neg R) \Rightarrow P \vee Q} \wedge L$$

$$\frac{\Rightarrow (P \vee R) \wedge (Q \vee \neg R) \rightarrow P \vee Q}{\Rightarrow (P \vee R) \wedge (Q \vee \neg R) \rightarrow P \vee Q} \rightarrow R$$

$$\frac{F, G, \Gamma \Rightarrow \Delta}{F \wedge G, \Gamma \Rightarrow \Delta} \wedge L$$

$$\frac{\frac{\overline{P \vee R, Q \vee \neg R \Rightarrow P \vee Q} \vee R}{(P \vee R) \wedge (Q \vee \neg R) \Rightarrow P \vee Q} \wedge L}{\Rightarrow (P \vee R) \wedge (Q \vee \neg R) \rightarrow P \vee Q} \rightarrow R$$

$$\frac{\Gamma \Rightarrow F, G, \Delta}{\Gamma \Rightarrow F \vee G, \Delta} \vee R$$

$$\frac{\frac{\frac{P \vee R, Q \vee \neg R \Rightarrow P, Q}{P \vee R, Q \vee \neg R \Rightarrow P \vee Q} \vee R}{(P \vee R) \wedge (Q \vee \neg R) \Rightarrow P \vee Q} \wedge L}{\Rightarrow (P \vee R) \wedge (Q \vee \neg R) \rightarrow P \vee Q} \rightarrow R$$

$$\frac{\Gamma \Rightarrow F, G, \Delta}{\Gamma \Rightarrow F \vee G, \Delta} \vee R$$

$$\begin{array}{c}
 \frac{P \vee R, Q \vee \neg R \Rightarrow P, Q}{P \vee R, Q \vee \neg R \Rightarrow P \vee Q} \vee R \\
 \frac{(P \vee R) \wedge (Q \vee \neg R) \Rightarrow P \vee Q}{\Rightarrow (P \vee R) \wedge (Q \vee \neg R) \rightarrow P \vee Q} \wedge L \\
 \Rightarrow (P \vee R) \wedge (Q \vee \neg R) \rightarrow P \vee Q \rightarrow R
 \end{array}$$

$$\frac{F, \Gamma \Rightarrow \Delta \quad G, \Gamma \Rightarrow \Delta}{F \vee G, \Gamma \Rightarrow \Delta} \vee L$$

$$\begin{array}{c}
\frac{P, Q \vee \neg R \Rightarrow P, Q}{\frac{P \vee R, Q \vee \neg R \Rightarrow P, Q}{\frac{P \vee R, Q \vee \neg R \Rightarrow P \vee Q}{(P \vee R) \wedge (Q \vee \neg R) \Rightarrow P \vee Q} \vee R} \vee L \\
\frac{\frac{P \vee R, Q \vee \neg R \Rightarrow P \vee Q}{(P \vee R) \wedge (Q \vee \neg R) \Rightarrow P \vee Q} \wedge L}{\Rightarrow (P \vee R) \wedge (Q \vee \neg R) \rightarrow P \vee Q} \rightarrow R
\end{array}$$

$$\frac{F, \Gamma \Rightarrow \Delta \quad G, \Gamma \Rightarrow \Delta}{F \vee G, \Gamma \Rightarrow \Delta} \vee L$$

$$\frac{\frac{\frac{\overline{P, Q \vee \neg R \Rightarrow P, Q} \text{ Ax} \quad \overline{R, Q \vee \neg R \Rightarrow P, Q}}{\vee L}}{\frac{P \vee R, Q \vee \neg R \Rightarrow P, Q}{\vee R}}}{\frac{(P \vee R) \wedge (Q \vee \neg R) \Rightarrow P \vee Q}{\wedge L}} \rightarrow R$$

$$\overline{A, \Gamma \Rightarrow A, \Delta} \text{ Ax}$$

$$\begin{array}{c}
\frac{P, Q \vee \neg R \Rightarrow P, Q \quad Ax}{\frac{P \vee R, Q \vee \neg R \Rightarrow P, Q}{P \vee R, Q \vee \neg R \Rightarrow P \vee Q} \vee R} \vee L \quad \frac{R, Q \vee \neg R \Rightarrow P, Q}{\frac{(P \vee R) \wedge (Q \vee \neg R) \Rightarrow P \vee Q} \wedge L} \rightarrow R} \vee L \\
\frac{F, \Gamma \Rightarrow \Delta \quad G, \Gamma \Rightarrow \Delta}{F \vee G, \Gamma \Rightarrow \Delta} \vee L
\end{array}$$

$$\begin{array}{c}
\frac{P, Q \vee \neg R \Rightarrow P, Q \quad Ax \quad \frac{\frac{R, Q \Rightarrow P, Q \quad R, \neg R \Rightarrow P, Q}{R, Q \vee \neg R \Rightarrow P, Q} \vee L}{P \vee R, Q \vee \neg R \Rightarrow P, Q} \vee L}{\frac{P \vee R, Q \vee \neg R \Rightarrow P, Q}{P \vee R, Q \vee \neg R \Rightarrow P \vee Q} \vee R} \wedge L \\
\frac{(P \vee R) \wedge (Q \vee \neg R) \Rightarrow P \vee Q}{\Rightarrow (P \vee R) \wedge (Q \vee \neg R) \rightarrow P \vee Q} \rightarrow R \\
\frac{F, \Gamma \Rightarrow \Delta \quad G, \Gamma \Rightarrow \Delta}{F \vee G, \Gamma \Rightarrow \Delta} \vee L
\end{array}$$

$$\begin{array}{c}
\frac{\overline{P, Q \vee \neg R \Rightarrow P, Q} \text{ Ax} \quad \frac{\overline{R, Q \Rightarrow P, Q} \text{ Ax} \quad \overline{R, \neg R \Rightarrow P, Q}}{R, Q \vee \neg R \Rightarrow P, Q} \vee L}{\overline{P \vee R, Q \vee \neg R \Rightarrow P, Q} \vee L} \vee L \\
\frac{\overline{P \vee R, Q \vee \neg R \Rightarrow P, Q}}{\overline{P \vee R, Q \vee \neg R \Rightarrow P \vee Q} \vee R} \vee R \\
\frac{\overline{(P \vee R) \wedge (Q \vee \neg R) \Rightarrow P \vee Q} \wedge L}{\overline{\Rightarrow (P \vee R) \wedge (Q \vee \neg R) \rightarrow P \vee Q} \rightarrow R} \rightarrow R
\end{array}$$

$$\overline{A, \Gamma \Rightarrow A, \Delta} \text{ Ax}$$

$$\begin{array}{c}
\frac{\overline{P, Q \vee \neg R \Rightarrow P, Q} \text{ Ax}}{\overline{P \vee R, Q \vee \neg R \Rightarrow P, Q} \text{ VL}} \quad \frac{\overline{R, Q \Rightarrow P, Q} \text{ Ax} \quad \frac{\overline{R \Rightarrow R, P, Q} \text{ Ax}}{\overline{R, \neg R \Rightarrow P, Q} \neg L}}{\overline{R, Q \vee \neg R \Rightarrow P, Q} \vee L} \\
\frac{\overline{P \vee R, Q \vee \neg R \Rightarrow P, Q} \text{ VL}}{\overline{P \vee R, Q \vee \neg R \Rightarrow P \vee Q} \vee R} \\
\frac{\overline{P \vee R, Q \vee \neg R \Rightarrow P \vee Q} \vee R}{\overline{(P \vee R) \wedge (Q \vee \neg R) \Rightarrow P \vee Q} \wedge L} \\
\frac{\overline{(P \vee R) \wedge (Q \vee \neg R) \Rightarrow P \vee Q} \wedge L}{\Rightarrow (P \vee R) \wedge (Q \vee \neg R) \rightarrow P \vee Q} \rightarrow R
\end{array}$$

$$\overline{A, \Gamma \Rightarrow A, \Delta} \quad \text{Ax}$$

Proof search properties

- ▶ For every logical operator (\neg etc) there is one left and one right rule
- ▶ Every formula in the premise of a rule is a subformula of the conclusion of the rule.
This is called the **subformula property**.
 \Rightarrow no need to guess anything when applying a rule backward
- ▶ Backward rule application terminates because one operator is removed in each step.

Instances of rules

Definition

An **instance** of a rule is the result of replacing Γ and Δ by multisets of concrete formulas and F and G by concrete formulas.

Example

$$\frac{\Rightarrow P \wedge Q, A, B}{\neg(P \wedge Q) \Rightarrow A, B}$$

is an instance of

$$\frac{\Gamma \Rightarrow F, \Delta}{\neg F, \Gamma \Rightarrow \Delta}$$

setting $F := P \wedge Q$, $\Gamma := \emptyset$, $\Delta := \{A, B\}$

Proof trees

Definition (Proof tree)

A **proof tree** is a tree whose nodes are sequents and where each parent-children fragment

$$\frac{S_1 \quad \dots \quad S_n}{S}$$

is an instance of a proof rule.

(\Rightarrow all leaves must be instances of axioms)

A sequent S is **provable** (or **derivable**) if there is a proof tree with root S .

We write $\vdash_G S$ to denote that S is derivable.

Proof trees

An alternative inductive definition of proof trees:

Definition (Proof tree)

If

$$\frac{S_1 \quad \dots \quad S_n}{S}$$

is an instance of a proof rule and

there are proof trees T_1, \dots, T_n with roots S_1, \dots, S_n

then

$$\frac{T_1 \quad \dots \quad T_n}{S}$$

is a proof tree (with root S).

What does $\Gamma \Rightarrow \Delta$ “mean”?

Definition

$$|\Gamma \Rightarrow \Delta| = \left(\bigwedge \Gamma \rightarrow \bigvee \Delta \right)$$

Example: $|\{A, B\} \Rightarrow \{P, Q\}| = (A \wedge B \rightarrow P \vee Q)$

Remember: $\bigwedge \emptyset = \top$ and $\bigvee \emptyset = \perp$

In the following slides we prove: $\vdash_G S$ iff $\models |S|$

Soundness

Lemma (Rule Equivalence)

For every rule
$$\frac{S_1 \quad \dots \quad S_n}{S}$$

- ▶ $|S| \equiv |S_1| \wedge \dots \wedge |S_n|$
- ▶ $|S|$ is a tautology iff all $|S_i|$ are tautologies

Proof: Exercise.

Theorem (Soundness of \vdash_G)

If $\vdash_G S$ then $\models |S|$.

Proof by induction on the height of the proof tree for $\vdash_G S$.

Tree must end in rule instance

$$\frac{S_1 \quad \dots \quad S_n}{S}$$

If $n = 0$ then we vacuously have $\models |S_i|$ for all i .

If $n > 0$ then by IH we also have $\models |S_i|$ for all i .

So $\models |S_i|$ for all i , hence $\models |S|$ by Rule Equivalence.

Proof search = growing a proof tree from the root

To prove completeness we first examine the properties of the **proof search procedure**:

- ▶ Start from an initial sequent S_0
- ▶ At each stage we have some potentially *partial* proof tree with unproved leaves
- ▶ In each step, pick some unproved leaf S and some rule instance

$$\frac{S_1 \quad \dots \quad S_n}{S}$$

and extend the tree with that rule instance
(creating new unproved leaves S_1, \dots, S_n)

Proof search terminates if ...

- ▶ there are no more unproved leaves — **success**
- ▶ there is some unproved leaf where no rule applies — **failure**
By the rules, **that leaf is of the form**

$$P_1, \dots, P_k \Rightarrow Q_1, \dots, Q_l$$

where all P_i and Q_j are atoms, no $P_i = Q_j$, and no $P_i = \perp$.

Example (failed proof)

$$\frac{\frac{\overline{P \Rightarrow P} \quad Ax \quad Q \Rightarrow P}{P \vee Q \Rightarrow P} \vee L \quad \frac{P \Rightarrow Q \quad \overline{Q \Rightarrow Q} \quad Ax}{P \vee Q \Rightarrow Q} \vee L}{P \vee Q \Rightarrow P \wedge Q} \wedge R$$

Falsifying assignments?

Proof search always terminates

Lemma (Termination)

Proof search terminates from any initial sequent S_0 .

Proof

In every step, one logical operator is removed.

- ⇒ Size of sequent decreases by 1
- ⇒ Depth of proof tree is bounded by size of S_0
- ⇒ Construction of proof tree terminates. □

Observe: Breadth only bounded by $2^{\text{size of } S_0}$.

Proof search preserves equivalence

Lemma (Search Equivalence)

At each stage of the search process,

if S_1, \dots, S_k are the unproved leaves, then $|S_0| \equiv |S_1| \wedge \dots \wedge |S_k|$

Proof by induction on the number of search steps.

Initially trivially true (base case).

When applying a rule instance

$$\frac{U_1 \quad \dots \quad U_n}{S_i}$$

we have

$$|S_0| \equiv |S_1| \wedge \dots \wedge |S_i| \wedge \dots \wedge |S_k|$$

(by IH)

$$\equiv |S_1| \wedge \dots \wedge |S_{i-1}| \wedge |U_1| \wedge \dots \wedge |U_n| \wedge |S_{i+1}| \wedge \dots \wedge |S_k|$$

(by Lemma Rule Equivalence)

Completeness

Lemma

If proof search fails, $|S_0|$ is not a tautology.

Proof If proof search fails, there is some unproved leaf

$$S = P_1, \dots, P_k \Rightarrow Q_1, \dots, Q_l$$

where all P_i, Q_j atoms, no $P_i = Q_j$ and no $P_i = \perp$.

Any assignment \mathcal{A} with $\mathcal{A}(P_i) = 1$ (for all i)

and $\mathcal{A}(Q_j) = 0$ (for all j) satisfies $\mathcal{A}(|S|) = 0$.

Thus $\mathcal{A}(|S_0|) = 0$ by Lemma Search Equivalence. □

Because of soundness of \vdash_G :

Corollary

Starting with some fixed S_0 , proof search cannot both fail (for some choices) and succeed (for other choices).

\Rightarrow no need for backtracking upon failure!

Completeness

Theorem (Completeness)

If $\models S$ then $\vdash_G S$.

Proof by contraposition: if not $\vdash_G S$ then proof search must fail.
Therefore $\not\models S$.

Corollary

Proof search is a decision procedure: it always terminates and it succeeds iff $\models S$.

Multisets versus sets

Termination only because of multisets.

With sets, the principal formula may get duplicated:

$$\frac{\Gamma \Rightarrow F, \Delta}{\neg F, \Gamma \Rightarrow \Delta} \quad \neg L \quad \Gamma := \{\neg F\} \rightsquigarrow \quad \frac{\neg F \Rightarrow F, \Delta}{\neg F \Rightarrow \Delta}$$

An alternative formulation of the set version:

$$\frac{\Gamma \setminus \{\neg F\} \Rightarrow F, \Delta}{\neg F, \Gamma \Rightarrow \Delta}$$

Gentzen used sequences (hence “sequent calculus”)

Admissible Rules and Cut Elimination

Admissible rules

Definition

A rule

$$\frac{S_1 \quad \dots \quad S_n}{S}$$

is **admissible** if $\vdash_G S_1, \dots, \vdash_G S_n$ together imply $\vdash_G S$.

\Rightarrow Admissible rules can be used in proofs like normal rules

Admissibility of

$$\frac{S_1 \quad \dots \quad S_n}{S}$$

can be shown semantically (using \vdash_G iff \models)

by proving that $\models |S_1|, \dots, \models |S_n|$ together imply $\models |S|$.

Proof theory is interested in **syntactic proofs** that show **how** to eliminate admissible rules.

Cut elimination rule

Theorem (Gentzen's *Hauptsatz*)

The cut elimination rule

$$\frac{\Gamma \Rightarrow F, \Delta \quad \Gamma, F \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \quad \textit{cut}$$

is admissible.

Proof Omitted.

Proofs with cut elimination can be much shorter than proofs without!

But: applying the rule needs creativity! (find the right F)

Intuitively: Proof of Gentzen's theorem shows how to replace creativity by calculation.

Many applications.