

Propositional Logic

Compactness

Compactness Theorem

Theorem

*A set S of formulas is satisfiable
iff every finite subset of S is satisfiable.*

Equivalent formulation:

*A set S of formulas is unsatisfiable
iff some finite subset of S is unsatisfiable.*

An application: Graph Coloring

Definition

A **4-coloring** of a graph (V, E) is a map $c : V \rightarrow \{1, 2, 3, 4\}$ such that $(x, y) \in E$ implies $c(x) \neq c(y)$.

Theorem (4CT)

A finite planar graph has a 4-coloring.

Theorem

A planar graph $G = (V, E)$ with countably many vertices $V = \{v_1, v_2, \dots\}$ has a 4-coloring.

Proof $G \rightsquigarrow$ set of formulas S s.t. S is sat. iff G is 4-col.

G is planar

$\Rightarrow$ every finite subgraph of G is planar and 4-col. (by 4CT)

$\Rightarrow$ every finite subset of S is sat.

$\Rightarrow S$ is sat. (by Compactness)

$\Rightarrow G$ is 4-col.

Proof details

$G \rightsquigarrow S$:

For simplicity:

atoms are of the form A_i^c where $c \in \{1, \dots, 4\}$ and $i \in \mathbb{N}$

$$\begin{aligned} S := & \{A_i^1 \vee A_i^2 \vee A_i^3 \vee A_i^4 \mid i \in \mathbb{N}\} \cup \\ & \{A_i^c \rightarrow \neg A_i^d \mid i \in \mathbb{N}, c, d \in \{1, \dots, 4\}, c \neq d\} \cup \\ & \{\neg(A_i^c \wedge A_j^c) \mid (v_i, v_j) \in E, c \in \{1, \dots, 4\}\} \end{aligned}$$

Subgraph corresponding to some $T \subseteq S$:

$$V_T := \{v_i \mid A_i^c \text{ occurs in } T \text{ (for some } c)\}$$

$$E_T := \{(v_i, v_j) \mid \neg(A_i^c \wedge A_j^c) \in T \text{ (for some } c)\}$$

Proof of Compactness

Theorem

*A set S of formulas is satisfiable
iff every finite subset of S is satisfiable.*

Proof

$\Rightarrow$: If S is satisfiable then every finite subset of S is satisfiable.

Trivial.

$\Leftarrow$: If every finite subset of S is satisfiable then S is satisfiable.

We prove that S has a model.

Proof of Compactness

Definition

Let $b_1 \cdots b_n \in \{0, 1\}^*$ with $n \geq 0$ and let T be a set of formulas. An assignment $\mathcal{A}$ is a $b_1 \cdots b_n$ -model of T if $\mathcal{A}(A_i) = b_i$ for every $i = 1, \dots, n$ and $\mathcal{A} \models T$.

In particular: every model is a ε -model.

Assume every finite $T \subseteq S$ is satisfiable. We prove:

1. There is an infinite sequence $b_1 b_2 \cdots \in \{0, 1\}^\omega$ such that for every $n \geq 1$ all finite $T \subseteq S$ have a $b_1 \cdots b_n$ -model.
2. The assignment $\mathcal{B}$ given by $\mathcal{B}(A_i) := b_i$ for all $i \geq 1$ is a model of S .

Proof of Compactness: Part (1)

To prove: There is an infinite sequence $b_1 b_2 \cdots \in \{0, 1\}^\omega$ such that for every $n \geq 1$ all finite $T \subseteq S$ have a $b_1 \cdots b_n$ -model.

It suffices to show:

- (a) Every finite $T \subseteq S$ has an ε -model.
- (b) For every sequence $\sigma \in \{0, 1\}^*$: if every finite $T \subseteq S$ has a σ -model then there exists $b \in \{0, 1\}$ such that every finite $T \subseteq S$ has a σb -model.

Proof of (a): By assumption every $T \subseteq S$ has an ε -model.

Proof of (b): Next slide.

Proof of Compactness: Part (1)

Proof of (b): By contradiction.

Assume that for some $\sigma \in \{0, 1\}$: every finite $T \subseteq S$ has a σ -model, but

(0) some finite $T_0 \subseteq S$ has no $\sigma 0$ -model; and

(1) some finite $T_1 \subseteq S$ has no $\sigma 1$ -model.

Consider the finite set $T_0 \cup T_1$.

By assumption, $T_0 \cup T_1$ has some σ -model $\mathcal{A}$. Let $n := |\sigma|$.

Two possible cases:

- ▶ $\mathcal{A}(A_{n+1}) = 0$. Then $\mathcal{A}$ is a $\sigma 0$ -model of T_0 , contradicting (0).
- ▶ $\mathcal{A}(A_{n+1}) = 1$. Then $\mathcal{A}$ is a $\sigma 1$ -model of T_1 , contradicting (1).

Proof of Compactness: Part (2)

To prove: The assignment $\mathcal{B}$ given by $\mathcal{B}(A_i) := b_i$ for all $i \geq 1$, where $b_1 b_2 \dots$ is the infinite sequence of (1), is a model of S .

We show $\mathcal{B} \models F$ for all $F \in S$.

Let m be the maximal index of all atoms in F .

By (1), $\{F\}$ has a $b_1 \dots b_m$ -model $\mathcal{A}$.

Hence $\mathcal{B} \models F$, because $\mathcal{A}$ and $\mathcal{B}$ agree on all atoms in F .

Corollary

Corollary

If $S \models F$ then there is a finite subset $M \subseteq S$ such that $M \models F$.