

# Propositional Logic

## Horn Formulas

# Efficient satisfiability checks

In this and the next slide sets:

- ▶ A very efficient satisfiability check for a special class of formulas in CNF: **Horn formulas**,
- ▶ Efficient satisfiability checks for arbitrary formulas in CNF: **DPLL** and **resolution** (later).

# Horn formulas

## Definition

A formula  $F$  in CNF is a **Horn formula** if every disjunction in  $F$  contains at most one positive literal.

Every disjunct of a Horn formula can equivalently be viewed as an implication  $K \rightarrow B$  where

- ▶  $K$  is a conjunction of atoms or  $\top$ , and
- ▶  $B$  is an atom or  $\perp$ .

$A$	$\equiv$	$(\top \rightarrow A)$	fact
$(\neg A \vee \neg B \vee C)$	$\equiv$	$(A \wedge B \rightarrow C)$	rule
$(\neg A \vee B)$	$\equiv$	$(A \rightarrow B)$	rule
$\neg A$	$\equiv$	$(A \rightarrow \perp)$	goal
$(\neg A \vee \neg B)$	$\equiv$	$(A \wedge B \rightarrow \perp)$	goal

## Satisfiability check for Horn formulas

Input: a Horn formula  $F$ .

Output: Model  $\mathcal{M}$  of  $F$  or “unsatisfiable”

```
for all atoms  $A_i$  in  $F$  do  $\mathcal{M}(A_i) := 0$ ;  
while  $F$  has a conjunct  $K \rightarrow B$   
    such that  $\mathcal{M}(K) = 1$  and  $\mathcal{M}(B) = 0$   
do  
    if  $B = \perp$  then return “unsatisfiable”  
    else  $\mathcal{M}(B) := 1$   
return  $\mathcal{M}$ 
```

Maximal number of iterations of the while loop:  
number of implications in  $F$

Each iteration requires at most  $O(|F|)$  steps.

Overall complexity:  $O(|F|^2)$

[Algorithm can be improved to  $O(|F|)$ . See Schöning.]

# Correctness of the model building algorithm

## Theorem

*The algorithm returns a model iff  $F$  is satisfiable.*

**Proof.** Invariant: if  $\mathcal{M}(A) = 1$ , then  $\mathcal{A}(A) = 1$  for every atom  $A$  and model  $\mathcal{A}$  of  $F$ .

(a) If “unsatisfiable” then unsatisfiable.

Assume  $F$  has model  $\mathcal{A}$  but algorithm answers “unsatisfiable”.

Let  $(A_{i_1} \wedge \dots \wedge A_{i_k} \rightarrow \perp)$  be the subformula causing “unsatisfiable”.

Since  $\mathcal{M}(A_{i_1}) = \dots = \mathcal{M}(A_{i_k}) = 1$ ,  $\mathcal{A}(A_{i_1}) = \dots = \mathcal{A}(A_{i_k}) = 1$ .

Then  $\mathcal{A}(A_{i_1} \wedge \dots \wedge A_{i_k} \rightarrow \perp) = 0$  and so  $\mathcal{A}(F) = 0$ , contradiction.

(b) If “ $\mathcal{M}$ ” then  $\mathcal{M} \models F$ .

After termination with “ $\mathcal{M}$ ”, every conjunct  $K \rightarrow B$  of  $F$  satisfies  $\mathcal{M}(K) = 0$  or  $\mathcal{M}(B) = 1$ .

Therefore  $\mathcal{M}(K \rightarrow B) = 1$  and thus  $\mathcal{M} \models F$ .

# Correctness of the model building algorithm

## Corollary

*A satisfiable Horn formula has a unique model with a smallest number of true atoms.*