

Propositional Logic
Definitional CNF
(Tseytin's transformation)

Definitional CNF

1. The **definitional CNF** of a formula is obtained in 2 steps:

Repeatedly replace a subformula G of the form $\neg A$, $A \wedge B$ or $A \vee B$ (A, B atoms!) by a new atom A' and conjoin $A' \leftrightarrow G$.

(This replacement is not applied to the “definitions” $A' \leftrightarrow G$ but only to the (remains of the) original formula.)

2. Translate all the subformulas $A' \leftrightarrow G$ into CNF.

Example

$$\begin{aligned} & \neg(\boxed{A_1 \vee A_2}) \wedge A_3 \\ \rightsquigarrow & \boxed{\neg A_4} \wedge A_3 \wedge (A_4 \leftrightarrow (A_1 \vee A_2)) \\ \rightsquigarrow & \boxed{A_5 \wedge A_3} \wedge (A_4 \leftrightarrow (A_1 \vee A_2)) \wedge (A_5 \leftrightarrow \neg A_4) \\ \rightsquigarrow & A_6 \wedge (A_4 \leftrightarrow (A_1 \vee A_2)) \wedge (A_5 \leftrightarrow \neg A_4) \wedge (A_6 \leftrightarrow (A_5 \wedge A_3)) \\ \rightsquigarrow & A_6 \wedge \text{CNF}(A_4 \leftrightarrow (A_1 \vee A_2)) \wedge \text{CNF}(A_5 \leftrightarrow \neg A_4) \wedge \text{CNF}(A_6 \leftrightarrow (A_5 \wedge A_3)) \end{aligned}$$

Definitional CNF: Complexity

Let the initial formula have size n .

1. Each replacement step increases the size of the formula by a constant.
There are at most as many replacement steps as subformulas, linearly many.
2. The conversion of each $A \leftrightarrow G$ into CNF increases the size by a constant.
There are only linearly many such subformulas.

Thus: the definitional CNF has size $O(n)$, and
can be constructed in $O(n)$ time.

Definitional CNF: Correctness — Notation

Definition

The notation $F[G/A]$ denotes the result of replacing **all** occurrences of the atom A in F by G .

We pronounce it as “ F with G for A ”.

Example

$$(A \wedge B)[(A \rightarrow B)/B] = (A \wedge (A \rightarrow B))$$

Definition

The notation $\mathcal{A}[v/A]$ denotes a modified version of $\mathcal{A}$ that maps A to v and behaves like $\mathcal{A}$ otherwise:

$$(\mathcal{A}[v/A])(A_i) = \begin{cases} v & \text{if } A_i = A \\ \mathcal{A}(A_i) & \text{otherwise} \end{cases}$$

Definitional CNF: Correctness — Substitution Lemma

Lemma

$\mathcal{A}(F[G/A]) = \mathcal{A}'(F)$ where $\mathcal{A}' = \mathcal{A}[\mathcal{A}(G)/A]$

Proof by structural induction on F .

► F is an atom:

If $F = A$: $\mathcal{A}(F[G/A]) = \mathcal{A}(G) = \mathcal{A}'(F)$

If $F \neq A$: $\mathcal{A}(F[G/A]) = \mathcal{A}(F) = \mathcal{A}'(F)$

► $F = F_1 \wedge F_2$:

$$\begin{aligned}\mathcal{A}((F_1 \wedge F_2)[G/A]) &= \mathcal{A}(F_1[G/A] \wedge F_2[G/A]) \\ &= \min(\mathcal{A}(F_1[G/A]), \mathcal{A}(F_2[G/A])) \\ &\stackrel{IH}{=} \min(\mathcal{A}'(F_1), \mathcal{A}'(F_2)) \\ &= \mathcal{A}'(F_1 \wedge F_2)\end{aligned}$$

Definitional CNF: Correctness

Each replacement step produces an equisatisfiable formula:

Lemma

Let A be an atom that does not occur in G .

Then $F[G/A]$ is equisatisfiable with $F \wedge (A \leftrightarrow G)$.

Proof Assume $\mathcal{A} \models F[G/A]$ for some assignment $\mathcal{A}$.

Let $\mathcal{A}' := \mathcal{A}[\mathcal{A}(G)/A]$. We prove $\mathcal{A}' \models F \wedge (A \leftrightarrow G)$.

$\mathcal{A}' \models F$: Substitution Lemma.

$\mathcal{A}' \models (A \leftrightarrow G)$: Because $\mathcal{A}'(A) = \mathcal{A}(G) = \mathcal{A}'(G)$
(by definition of $\mathcal{A}'$ and because A does not occur in G).

Assume $\mathcal{A} \models F \wedge (A \leftrightarrow G)$ for some assignment $\mathcal{A}$.

We prove $\mathcal{A} \models F[G/A]$, that is, $\mathcal{A}(F[G/A]) = 1$.

We show $\mathcal{A}(F[G/A]) = \mathcal{A}'(F) = \mathcal{A}(F) = 1$ for $\mathcal{A}' := \mathcal{A}[\mathcal{A}(G)/A]$.

$\mathcal{A}(F[G/A]) = \mathcal{A}'(F)$: Substitution Lemma.

$\mathcal{A}'(F) = \mathcal{A}(F)$: From $\mathcal{A} \models (A \leftrightarrow G)$ follows $\mathcal{A}(A) = \mathcal{A}(G)$, and
so $\mathcal{A}' = \mathcal{A}$.

$\mathcal{A}(F) = 1$: Because $\mathcal{A} \models F$.

Definitional CNF: Correctness

Does $F \wedge (A \leftrightarrow G) \models F[G/A]$ hold?

Does $F[G/A] \models F \wedge (A \leftrightarrow G)$ hold?

Summary

Theorem

For every formula F of size n
there is an *equisatisfiable CNF* formula G of size $O(n)$.

Proof.

Repeated application of the Lemma.



Similarly it can be shown:

Theorem

For every formula F of size n
there is an *equivalent DNF* formula G of size $O(n)$.

Validity of CNF

Validity of formulas in CNF can be checked in linear time.

A formula in CNF is valid iff all its disjunctions are valid.

A disjunction is valid iff it contains both an atomic A and $\neg A$ as literals.

Example

Valid: $(A \vee \neg A \vee B) \wedge (C \vee \neg C)$

Not valid: $(A \vee \neg A) \wedge (\neg A \vee C)$

Satisfiability of DNF

Satisfiability of formulas in DNF can be checked in linear time.

A formula in DNF is satisfiable iff at least one of its conjunctions is satisfiable. A conjunction is satisfiable iff it does not contain both an atomic A and $\neg A$ as literals.

Example

Satisfiable: $(\neg B \wedge A \wedge B) \vee (\neg A \wedge C)$

Unsatisfiable: $(A \wedge \neg A \wedge B) \vee (C \wedge \neg C)$

Satisfiability/validity of DNF and CNF

Theorem

*Satisfiability of formulas in **CNF** is NP-complete.*

Theorem

*Validity of formulas in **DNF** is co-NP-complete.*

Standard decision procedure for validity of F :

1. Transform $\neg F$ into an equisat. formula G in def. CNF
2. Apply efficient CNF-based SAT solver to G