Exercise sheet: Resolution and Sequent Calculus

Exercise 1: Input resolution

Let $\mathbb{C} = \{C_1, \ldots, C_n\}$ be a set of clauses. We say a sequence $\langle B_0, B_1, \ldots, B_k\rangle$ is an *input resolution* of B_k from $\mathbb{C}$ if the following holds:

(i) $B_0 \in \mathbb{C}$, and

(ii) $B_{\ell+1}$ is the resolvent of B_ℓ and one clause from $\mathbb{C}$.

1. Show that there is a resolution of $\Box$ but no input resolution of $\Box$ from
$$F = \{\{\neg z, x\}, \{\neg x, \neg y\}, \{y, z\}, \{z, \neg y, x\}, \{y, \neg x\}\}.$$
2. Show that there is an input resolution of $\Box$ from
$$\{\{\neg t, \neg y\}, \{\neg y, z\}, \{\neg x, \neg z, t\}, \{\neg x, y\}, \{x\}\}.$$
3. Prove that input resolution is complete for Horn formulae by showing that there is an input resolution of $\Box$ from any unsatisfiable set of clauses where every clause contains at most one positive literal.[1]

   Hint: Consider a Horn formula as a conjunction of *rules* $\bigwedge A \to H$ with A being a set of propositional variables and H being either a propositional variable or 0, i.e. the *procedural reading* of Horn formulae. Try to consider an "inverse" marking algorithm which starts from 0 and marks propositional variables on the left hand side of implications until a proof why 0 is implied by any possible assignment which sets the right hand sides of rules $1 \to H \equiv \bigwedge \emptyset \to H$ to true is found. Then, frame the executed steps in terms of input resolution.

Exercise 2: Positive Resolution

We call a resolution step $C_1, C_2 \vdash_{Res} R$ positive or a *P-Resolution* if one of the clauses C_1, C_2 contains only possitive literals. Prove that resolution restricted to positive resolution steps is complete.

Exercise 3: Currying

Prove that for any $n \geq 1$, the following formula has a sequent calculus proof:
$$(A_1 \wedge (A_2 \wedge (\cdots \wedge A_n) \cdots) \to B) \to (A_1 \to A_2 \to \cdots \to A_n \to B)$$
Remember that $A \to B \to C = A \to (B \to C)$

[1] Note that input resolution is not complete in general because F from 1. is unsatisfiable but there is no input resolution proof of that fact.

Exercise 4: Bonus: Sequent Calculus Proof Search

Write a tool that given a formula of propositional logic searches for a sequent calculus proof.

You can use the following tool to verify the output: `https://github.com/astrobeastie/seqc`. If you write your solution in rust you can directly integrate your proof search. Otherwise it can read proofs from files. Note that seqc was written quite hastily and likely still contains a few bugs. If you find one open an issue or message me on zulip.