

EXERCISE SHEET: PROPOSITIONAL LOGIC

Exercise 1: CNF Conversion

- (a) Prove that converting a formula into Negation Normal Form (NNF) terminates, by giving a weight function $w: \text{formula} \rightarrow \mathbb{N}$ such that the following inequalities hold for all F, G :

$$\begin{aligned}w(\neg\neg F) &> w(F) \\w(\neg(F \vee G)) &> w(\neg F \wedge \neg G) \\w(\neg(F \wedge G)) &> w(\neg F \vee \neg G)\end{aligned}$$

$w(F)$ should be defined recursively using only addition, subtraction, multiplication, division and exponentiation.

Note: By “defined recursively” we mean that there are functions $f_{\neg}: \mathbb{N} \rightarrow \mathbb{N}$, $f_{\vee}: \mathbb{N}^2 \rightarrow \mathbb{N}$, $f_{\wedge}: \mathbb{N}^2 \rightarrow \mathbb{N}$ such that $w(\neg F) = f_{\neg}(w(F))$, $w(F \vee G) = f_{\vee}(w(F), w(G))$, and $w(F \wedge G) = f_{\wedge}(w(F), w(G))$.

- (b) Prove that the second step of converting a Formula to CNF terminates, by giving a weight function $w: \text{formula} \rightarrow \mathbb{N}$ such that the following inequalities hold for all F, G, H :

$$\begin{aligned}w(F \vee (G \wedge H)) &> w((F \vee G) \wedge (F \vee H)) \\w((F \wedge G) \vee H) &> w((F \vee H) \wedge (G \vee H))\end{aligned}$$

$w(F)$ should be defined recursively using only addition, subtraction, multiplication, division and exponentiation.

- (c) **Challenge:** find a weight function that fulfills the requirements of both (a) and (b).

Exercise 2: Large disjunctive normal form

- Write down a **DNF**-formula equivalent to $(a_1 \vee b_1) \wedge (a_2 \vee b_2) \wedge \cdots \wedge (a_n \vee b_n)$.
- Prove that any **DNF**-formula equivalent to the above formula must have at least 2^n clauses.

Exercise 3: Perfect matching

A **perfect matching** in an undirected graph $G = (V, E)$ is a subset of the edges $M \subseteq E$ such that every vertex $v \in V$ is an endpoint of exactly one edge in M . Given a finite graph G , describe how to obtain a propositional formula F_G such that F_G is satisfiable if and only if G has a perfect matching. The formula F_G should be computable from G in time polynomial in $|V|$.