Chair for Foundations of Software Reliability and Theoretical Computer Science
School of Computation, Information and Technology
Technical University of Munich

Eexam
Place student sticker here

Note:

- During the attendance check a sticker containing a unique code will be put on this exam.
- This code contains a unique number that associates this exam with your registration number.
- This number is printed both next to the code and to the signature field in the attendance check list.

Automata and Formal Languages

Exam:	IN2041 / Endterm	**Date:**	Thursday 12th February, 2026
Examiner:	Prof. Dr. Javier Esparza	**Time:**	11:00 – 13:00

Working instructions

- This exam consists of **12 pages** with a total of **7 problems**.
  Please make sure now that you received a complete copy of the exam.
- The total amount of achievable credits in this exam is 75 credits, including 5 bonus credits.
- To pass the exam, 35 credits are *sufficient*.
- Detaching pages from the exam is prohibited.
- Allowed resources:
  - one **analog dictionary** English $\leftrightarrow$ native language
- Subproblems marked by * can be solved without results of previous subproblems.
- **Answers are only accepted if the solution approach is documented.** Give a reason for each answer unless explicitly stated otherwise in the respective subproblem.
- Do not write with red or green colors nor use pencils.
- Physically turn off all electronic devices, put them into your bag and close the bag.

Left room from ________ to ________	/	Early submission at ________	

Problem 1 **Quiz (15 credits)**

For each of these statements, decide whether it is true or false. If it is true, give a proof; if it is false, give a counterexample. Otherwise no points will be awarded! We use $\Sigma := \{a, b\}$ as alphabet in this exercise.

a)* Let A be an NFA such that all states of A are reachable. True or false: The output of $NFAtoDFA(A)$ has at least as many states as A.

b)* Let A be an NFA with n states. True or false: There exists an NBA for $(L(A))^\omega$ with $n+2$ states.

c)* Let A be a minimal NBA (that is, no NBA B with $L(B) = L(A)$ has less states than A). True or false: Every state of A recognizes a different ω-language.

d)* Let L be an ω-regular language. True or false: There exists an NRA with only one Rabin pair recognizing L.

0
1
2

e)* Let q, r be two reachable states of an NBA A such that r is reachable from q, but q is not reachable from r. Consider a run of *NestedDFS* on A. True or false: $d[r] < f[q]$, i.e. the algorithm discovers r before it backtracks from q. If the answer is "true", prove the statement using the parenthesis theorem given below; if the answer is "false", give a counterexample.

Parenthesis theorem. Let $I(q)$ denote the interval $[d[q], f[q]]$, and let $I(q) \prec I(r)$ denote that $f[q] < d[r]$ holds. In a DFS-tree, for any two states q and r, one of the following four conditions holds: (1) $I(q) \subseteq I(r)$ and q is a descendant of r; (2) $I(r) \subseteq I(q)$ and r is a descendant of q; (3) $I(q) \prec I(r)$ and neither is a descendant of the other; (4) $I(r) \prec I(q)$ and neither is a descendant of the other.

0
1
2
3

f)* Let AP = $\{p, q, r\}$ be a set of atomic propositions and let $\sigma \in (2^{\mathrm{AP}})^\omega$ be a computation. True or false: If $\sigma \models p \ \mathbf{U}\ (q \ \mathbf{U}\ r)$, then $\sigma \models (p \ \mathbf{U}\ q) \ \mathbf{U}\ r$.

0
1
2

Problem 2 **Regular languages (7 credits)**

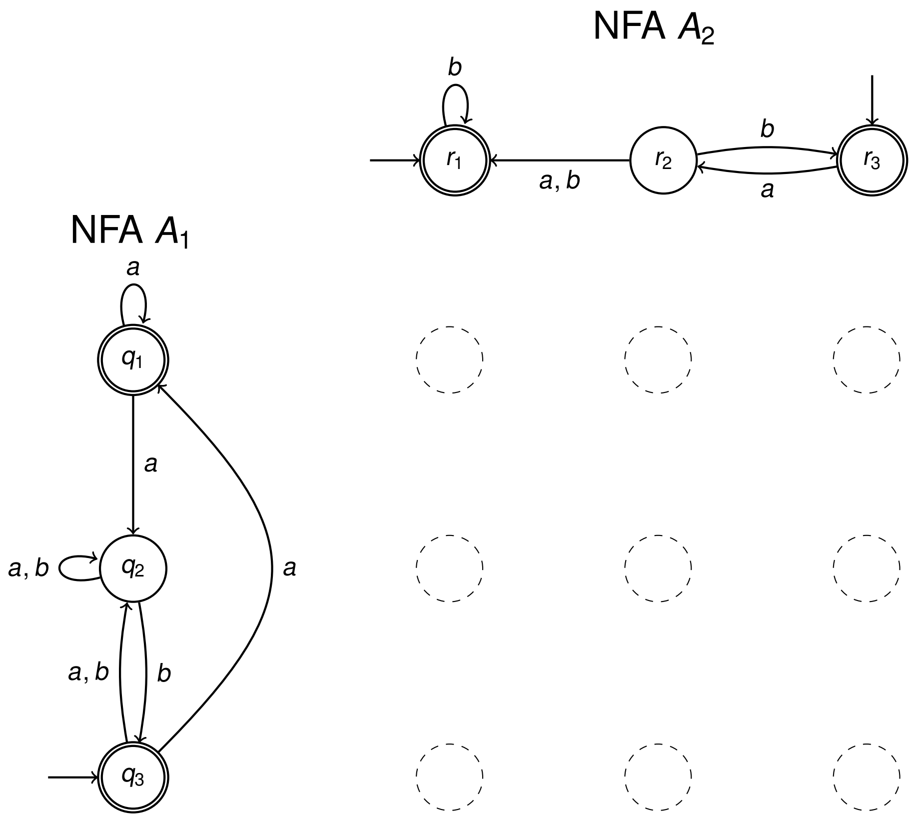
For $n \geq 1$, we define $\Sigma_n := \{1, ..., n\}$ and $L_n := \{w \in \Sigma_n^* \mid \text{at least one letter of } \Sigma_n \text{ does not appear in } w\}$. For example, we have $\varepsilon, 1111, 23 \in L_3$ and $123, 2231 \notin L_3$.

a)* Sketch an NFA for L_n with $\mathcal{O}(n)$ states.

b)* **Bonus points:** Show using residuals that every DFA for L_n has at least 2^n states.

Problem 3 **Product construction (11 credits)**

a)* Compute an NFA for $L(A_1) \cap L(A_2)$ using the product construction. You only need to construct reachable states and transitions.



Now consider the NGA (generalized Büchi automaton) B_1 and the NBA B_2 defined as follows:

- B_1 has the same states, transitions, and initial states as A_1, and acceptance condition $\{\{q_1\}, \{q_3\}\}$ (that is, B_1 has two sets of accepting states, $\{q_1\}$ and $\{q_3\}$).
- B_2 is just A_2, but interpreted as a Büchi automaton; in particular, the set of accepting states of B_2 is $\{r_1, r_3\}$.

b)* Consider the NGA B with $L_\omega(B) = L_\omega(B_1) \cap L_\omega(B_2)$ which results from using the operation for intersection of NGAs described in the course. What is the acceptance condition of B?

c)* Give a minimal NGA C accepting the same language as the NGA B from (b). Justify your answer.

Problem 4 **Pattern matching (6 credits)**

Consider the pattern $p = bbbabb$ over the alphabet $\Sigma = \{a, b\}$.

0
1

a)* Construct the NFA A_p recognizing $\Sigma^* p$ according to the construction specified in the course.

0
1
2

b) Construct the DFA B_p by applying the powerset construction to A_p.

0
1
2
3

c) Construct the lazy DFA C_p for the pattern p by using B_p.

Problem 5 **Transducers (8 credits)**

For this exercise, let $\Sigma = \{a, b\}$.

a)* Is there a transducer for the relation $R_1 := \{(a_1 \cdots a_n, a_n \cdots a_1) \mid n \in \mathbb{N},\ a_1, ..., a_n \in \Sigma\}$? If yes, draw one. If no, prove it using residuals.

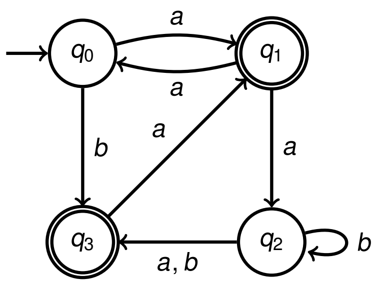
0 1 2 3 4

b)* Is there a transducer for the relation $R_2 := \{(a_1 \cdots a_n, a_n a_1 \cdots a_{n-1}) \mid n \in \mathbb{N},\ a_1, ..., a_n \in \Sigma\}$? If yes, draw one. If no, prove it using residuals.

0 1 2 3 4

Problem 6 Büchi Complementation (10 credits)

Consider the following Büchi automaton $\mathcal{B}$:



We denote by $\overline{\mathcal{B}}$ the complement of $\mathcal{B}$, defined using rankings and owing states as in the lecture. We write the states of $\overline{\mathcal{B}}$ as pairs $[r, O]$ where r is a ranking and O is a set of owing states. In r, we write the ranks of the nodes in the order q_0, q_1, q_2, q_3 from top to bottom.

a)* For each of the following pairs $[r, O]$, decide whether it is a state of $\overline{\mathcal{B}}$. If it is not, give a justification.
Note: The question is whether the pairs are states, not whether they are reachable from the initial states.

1. $\left[\begin{array}{c}\bot\\2\\5\\0\end{array}, \{q_0, q_1, q_3\}\right]$

2. $\left[\begin{array}{c}6\\0\\7\\\bot\end{array}, \emptyset\right]$

3. $\left[\begin{array}{c}5\\5\\\bot\\2\end{array}, \{q_3\}\right]$

4. $\left[\begin{array}{c}1\\2\\2\\0\end{array}, \{q_0, q_1\}\right]$

b)* For each of the following transitions, decide whether it is a transition of $\overline{\mathcal{B}}$. If it is not, give a justification.
Note: The question is whether these are transitions, not whether they are reachable transitions.

1. $\left[\begin{matrix}\bot\\0\\\bot\\4\end{matrix}, \{q_1, q_3\}\right] \xrightarrow{a} \left[\begin{matrix}0\\2\\\bot\\\bot\end{matrix}, \{q_0, q_1\}\right]$

2. $\left[\begin{matrix}4\\4\\\bot\\\bot\end{matrix}, \{q_1\}\right] \xrightarrow{a} \left[\begin{matrix}3\\2\\3\\\bot\end{matrix}, \{q_1\}\right]$

3. $\left[\begin{matrix}1\\\bot\\6\\4\end{matrix}, \{q_3\}\right] \xrightarrow{b} \left[\begin{matrix}\bot\\\bot\\6\\4\end{matrix}, \emptyset\right]$

4. $\left[\begin{matrix}2\\4\\\bot\\\bot\end{matrix}, \{q_0, q_1\}\right] \xrightarrow{a} \left[\begin{matrix}2\\0\\2\\\bot\end{matrix}, \{q_0, q_1\}\right]$

5. $\left[\begin{matrix}3\\4\\5\\\bot\end{matrix}, \{q_1\}\right] \xrightarrow{a} \left[\begin{matrix}4\\2\\1\\0\end{matrix}, \{q_0\}\right]$

6. $\left[\begin{matrix}3\\\bot\\4\\0\end{matrix}, \emptyset\right] \xrightarrow{b} \left[\begin{matrix}1\\\bot\\3\\2\end{matrix}, \{q_3\}\right]$

Problem 7 Construction (18 credits)

Answer the following questions. No justification is needed.

0 1 2 3

a)* Consider the regular expression $r = a^*b^*c^* + c^*a^*$ over the alphabet $\{a, b, c\}$. Give regular expressions for *all* residuals of $L(r)$.

0 1 2

b)* Let $L := \{aab, abb, bab, bbb\}$. Give the kernel $\langle L \rangle$ of L.

0 1 2 3 4

c)* Give formulas of first-order logic on words for the following languages over the alphabet $\{a, b, c\}$. You may use the macros *first*(x), *last*(x) and $x = y + 1$, $x = y + 2$, ... without definition. If you use other macros, you must define them first.

1. $L(a^*ba^*c)$
2. $L((a + b)^*cc)$
3. $L((ab)^*)$

0 1 2 3 4

d)* Give a *deterministic* Büchi automaton over the alphabet $\{a, b, c\}$ for the language of the ω-regular expression $(ab)^*c^\omega + (ca)^*b^\omega$.

e)* Let $\Sigma = \{a, b\}$. We say that a letter is *isolated* if it is neither preceded nor followed by the same letter. For example, in the word $aaa\mathbf{b}aa\mathbf{ba}b^\omega$ the three highlighted letters are isolated. Let $L \subseteq \Sigma^\omega$ be the language of ω-words where infinitely many letters are isolated. Give an NBA for L.

0 1 2 3

f)* Let $\mathrm{AP} = \{p, q\}$. Give an ω-regular expression for all computations over $\Sigma = 2^{\mathrm{AP}}$ satisfying $(\mathbf{G}p)\ \mathbf{U}\ q$.

0 1 2

Additional space for solutions–clearly mark the (sub)problem your answers are related to and strike out invalid solutions.