Chair for Foundations of Software Reliability and Theoretical Computer Science
School of Computation, Information and Technology
Technical University of Munich

Esolution
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Note:

- During the attendance check a sticker containing a unique code will be put on this exam.
- This code contains a unique number that associates this exam with your registration number.
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Petrinetze

Exam:	IN2052 / Endterm	**Date:**	Thursday 1st August, 2024
Examiner:	Prof. Javier Esparza	**Time:**	11:30 – 12:45

	P 1	P 2	P 3
I			
II			

Left room from ______________ to ______________

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Early submission at ______________

Notes ____

Chair for Foundations of Software Reliability and Theoretical Computer Science
School of Computation, Information and Technology
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Endterm

Petrinetze

Prof. Javier Esparza
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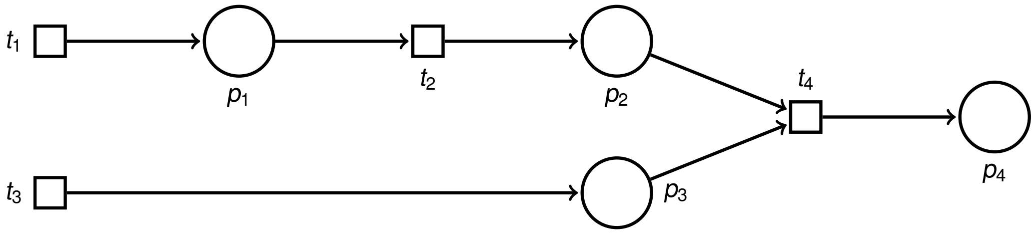
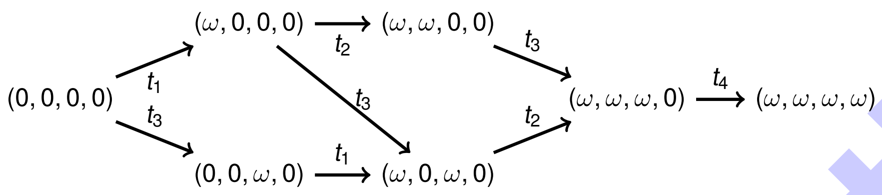
Working instructions

- This exam consists of **4 pages** with a total of **3 problems**.
  Please make sure now that you received a complete copy of the exam.
- The total amount of achievable credits in this exam is 43 credits.
- Detaching pages from the exam is prohibited.
- Allowed resources:
  - one **non-programmable pocket calculator**
  - one **analog dictionary** English ↔ native language
- Subproblems marked by * can be solved without results of previous subproblems.
- **Answers are only accepted if the solution approach is documented.** Give a reason for each answer unless explicitly stated otherwise in the respective subproblem.
- Do not write with red or green colors nor use pencils.
- Physically turn off all electronic devices, put them into your bag and close the bag.

Problem 1 Coverability Graphs (10 credits)

a)* Construct the coverability graph of the following Petri net $(\mathcal{N}, M_0)$, where $M_0 = \mathbf{0}$, that is, the initial marking puts zero tokens in all places.



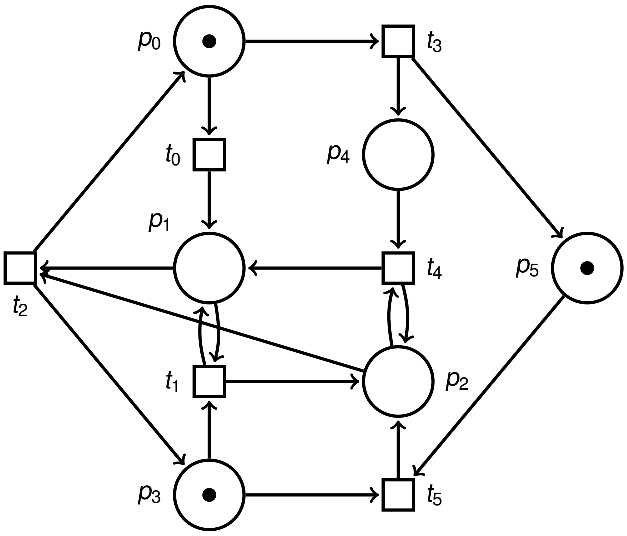


b)* Prove that the coverability graph of a Petri net $(\mathcal{N}, M_0)$ with $M_0 = \mathbf{0}$ contains at most 2^n nodes, where n is the number of places of $\mathcal{N}$.

We will prove that all nodes v of the coverability graph are labelled with 0 and ω only. Then the given bound follows immediately.
We will prove this by induction on the distance d of v from the root node labeled $M_0 = \mathbf{0}$. For distance $d = 0$ we have $v = M_0$, which fulfills this.
$d \to d + 1$: Let v be a node at distance $d + 1$. Then there exists a node w at distance d from the root, and such that $w \to v$ via a transition t. By induction hypothesis, w is labelled with 0 and ω only.
Since t is enabled at w, no place $p \in {}^\bullet t$ is unmarked at w. Then it follows that every $p \in {}^\bullet t$ is marked with ω. It follows that t does not decrease any places $(\omega - 1 = \omega)$, i.e. we have $v \geq w$. This means that every place $p \in t^\bullet$ will be marked with ω in v. Places which are not in $t^\bullet$ do not change their marking, hence v again only has 0 and ω as values.

Problem 2 Analyzing a Petri Net (15 credits)

Consider the following net $\mathcal{N}$:



Answer the following questions about the Petri net $\mathcal{N}$.
If the object exists, it is enough to give one, otherwise prove that such an object cannot exist.

a)* Is there an S-invariant I with $I(p_0) > 0$?

☒ True ☐ False

Yes, $I(p_0) = I(p_1) = I(p_4) = 1$, with $I(p) = 0$ for all other p.

b)* Is there an S-invariant I with $I(p_5) > 0$?

☐ True ☒ False

No, from t_5 and t_1 we obtain $I(p_5) + I(p_3) = I(p_2)$ and $I(p_3) = I(p_2)$ respectively.
We can subtract these equations to obtain $I(p_5) = 0$.

c)* Is there a positive T-invariant?

☒ True ☐ False

Yes, both the parikh mappings of $t_3t_4t_5t_2$ and $t_0t_1t_2$ are T-invariants.
Therefore their sum, which is positive on every coordinate, is again a T-invariant.

d)* Give a smallest siphon containing p_5. (That is, a siphon containing p_5 that does not properly contain any other siphon P'' with $p_5 \in P''$.)

Because of t_3, we have to contain p_0. Because of t_2 we then contain either p_1 or p_2. A further case distinction gives the possibilities $\{p_0, p_1, p_2, p_5\}$, $\{p_0, p_1, p_4, p_5\}$ and $\{p_0, p_2, p_3, p_5\}$.

e)* Fact: Every minimal proper siphon of $(\mathcal{N}, M_0)$ is an initially marked trap. (You do not have to prove this fact.) Using this fact, prove that $(\mathcal{N}, M_0)$ is deadlock-free.

By the lecture, if every proper siphon contains an initially marked trap, then the net is deadlock-free. Every proper siphon contains a minimal proper siphon, which by the fact is an initially marked trap.

Problem 3 Reductions (18 credits)

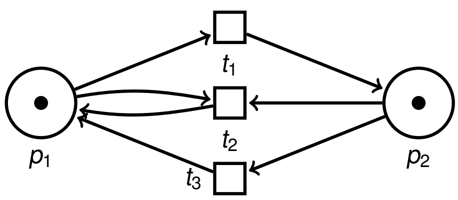
a)* Let $\mathcal{N} = (S, T, F)$ be a net with $T = \{t_1, \dots, t_m\}$. A sequence $\sigma \in T^*$ is *ordered* if σ belongs to the language of the regular expression $t_1^* t_2^* \cdots t_m^*$. For example, $t_2 t_3$ and $t_1 t_1 t_2 t_4 t_4 t_4$ are ordered, but $t_2 t_1$ is not ordered.
The *ordered* reachability problem is defined as follows: Given a net $\mathcal{N} = (S, T, F)$ with a set $T = \{t_1, \dots, t_m\}$ of transitions, and two markings M_0, M of $\mathcal{N}$, is there an ordered firing sequence σ such that $M_0 \xrightarrow{\sigma} M$?
Prove that the ordered reachability problem can be reduced in polynomial time to the reachability problem. Describe the reduction in detail, and *sketch* the proof that the reduction is correct.

Let $\mathcal{N} = (S, T, F)$ be a net with $T = \{t_1, \dots, t_m\}$, and M_0, M two markings.
We create a Petri net $\mathcal{N}' = (S', T', F')$ and new markings M_0', M' as follows:
The idea is to have m "phases" p_i', such that in the i-th phase only the i-th transition can be used.
Formally, we define $S' := S \cup \{p_1', \dots, p_m'\}$, where $p_1', \dots, p_m'$ are new places, i.e. $S \cap \{p_1', \dots, p_m'\} = \emptyset$.
We define $T' := T \cup \{t_1', \dots, t_{m-1}'\}$, where again $t_1', \dots, t_{m-1}'$ are new transitions, i.e. $T \cap \{t_1', \dots, t_{m-1}'\} = \emptyset$.
We define $F' := F \cup \{(p_i', t_i), (t_i, p_i') \mid 1 \le i \le m\} \cup \{(p_i', t_i'), (t_i', p_{i+1}') \mid 1 \le i \le m-1\}$.
We finish by defining $M_0' := M_0 + \{p_1'\}$ and $M' := M + \{p_m'\}$.
Correctness Sketch: $\Rightarrow$: To simulate an ordered transition sequence $t_1^{r_1} t_2^{r_2} \dots t_m^{r_m}$ from $M_0 \rightarrow_{\mathcal{N}}^* M$, we can fire $t_1^{r_1} t_1' t_2^{r_2} t_2' \dots t_{m-1}' t_m^{r_m}$ in $\mathcal{N}'$. Observe in particular that the new arcs to and from the t_i do not change the effect of t_i, they only cause it to be firable "in phase i only".
$\Leftarrow$: On the other hand, by construction of $\mathcal{N}'$, any transition sequence of $\mathcal{N}'$ ending with a token at p_m' has to be of the form $t_1^{r_1} t_1' t_2^{r_2} t_2' \dots t_{m-1}' t_m^{r_m}$ for some numbers $r_i \in \mathbb{N}$, from which we can read of a firing sequence $t_1^{r_1} t_2^{r_2} \dots t_m^{r_m}$ of $\mathcal{N}$.

b)* Let $\mathcal{N}$ be a net without weights and let M_0, M be two markings of $\mathcal{N}$ satisfying $M_0, M \ge \mathbf{1}$, that is, satisfying that $M_0(s) \ge 1$ and $M(s) \ge 1$ hold for every place s of $\mathcal{N}$.
Prove: If the marking equation $M = M_0 + N \cdot X$ has a nonnegative integer (i.e. $\mathbb{N}$) solution X, then there exists a natural number $\lambda \in \mathbb{N}_{>0}$ such that λM is reachable from λM_0.
Hint: Let σ have parikh vector X, and prove $i \cdot M_0 + (\lambda - i)M \xrightarrow{\sigma} (i-1) \cdot M_0 + (\lambda - i + 1)M$ for all $1 \le i \le \lambda$.

Define $\lambda := ||X||_1$ to be the number of transitions of X. We claim $\lambda M_0 \rightarrow^* \lambda M$.
To prove the claim, let σ be any transition sequence with parikh vector X. It suffices to show that for all $1 \le i \le \lambda$ we have $i \cdot M_0 + (\lambda - i)M \xrightarrow{\sigma} (i-1)M_0 + (\lambda - i + 1)M$.
To see this, observe that $i \cdot M_0 + (\lambda - i)M \ge i \cdot \mathbf{1} + (\lambda - i) \cdot \mathbf{1} = \lambda \cdot \mathbf{1} = ||X||_1 \cdot \mathbf{1}$. Since no transition consumes more than 1 token from any place, and σ only contains $||X||_1$ many transitions, we do not run out of tokens on any place when trying to fire σ. Hence σ is firable at $i \cdot M_0 + (\lambda - i)M$, and since its parikh vector X is a solution to the marking equation, it leads to the claimed marking.

c)* Prove that the assumption $M \ge \mathbf{1}$ is necessary, by giving a net $\mathcal{N}$ without weights and two markings M_0, M such that $M_0 \ge \mathbf{1}$ and the marking equation $M = M_0 + N \cdot X$ has a nonnegative integer solution, but there does not exist a natural number $\lambda \in \mathbb{N}_{>0}$ such that λM is reachable from λM_0.



We let $M := \mathbf{0}$. The marking equation has the solution $t_1 \mapsto 1, t_2 \mapsto 2, t_3 \mapsto 0$. However, $P = \{p_1, p_2\}$ is a trap, hence $\lambda M = M$ cannot be reachable from any non-zero marking.