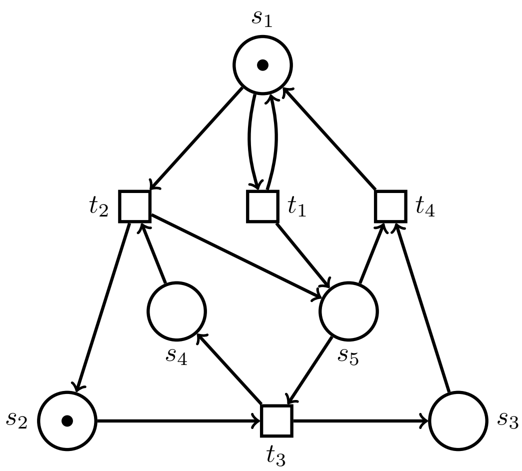
Petri nets — Retake

- You have **75 minutes** to complete the exam.
- Write your name, Matrikelnummer (immatriculation number) and page number on every sheet.
- Write with a black or blue **pen**. Do not use red or green.
- You can obtain **36 points**. You need **15 points** to pass.

Except for the second question, the questions are over Petri nets that are connected, and that do not have weights on the arcs. The symbol ★ denotes that we consider this question to be a bit harder.

Question 1 $(3+3+6=12$ points)
Consider the following Petri net (N, M_0):



(a) Draw the coverability graph of (N, M_0).

(b) Is (N, M_0) deadlock free? If so, give a proof; if not, give a firing sequence leading to a dead marking.

(c) ★ Show that if M is any marking such that $M(s_1) > 0$ and $M(s_4) > 0$, then (N, M) is **not** live. **Hint:** The set $\{s_1, s_5\}$ is a siphon.

Question 2 $(3+3+3=9$ points)
We consider nets with four distinguished places called `Start`, `Input`, `End` and `Output`. The nets may also have other places. Given such a net N and some $n \geq 0$, we denote by $\texttt{Start}_n$ the marking that puts

1 token in `Start` n tokens in `Input` **and 0 tokens elsewhere.**

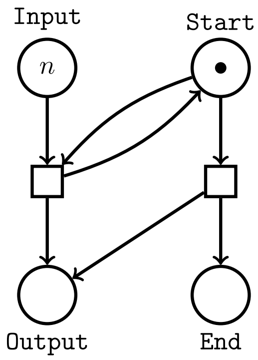
Similarly, we denote by $\texttt{End}_n$ the marking that puts

1 token in `End` n tokens in `Output` **and 0 tokens elsewhere.**

We say N **weakly computes** a function $f(n)$ if for every $n \in \mathbb{N}$ (where $\mathbb{N}$ contains 0) the following two conditions hold:

- $\texttt{Start}_n$ can reach $\texttt{End}_{f(n)}$.
- $\texttt{Start}_n$ **cannot reach** $\texttt{End}_m$ for any $m > f(n)$.

For example, the following net weakly computes the function $f(x) = x + 1$.



Give Petri nets **with weighted arcs** which weakly compute the following functions:

(a) the function

$$f(x) = \begin{cases} 1 & \text{if } 2 \leq x \\ 0 & \text{otherwise} \end{cases}$$

The Petri net you give must have at most 5 places.

(b) the function

$$f(x) = \begin{cases} 1 & \text{if } x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

The Petri net you give must have at most 5 places.

(c) the function

$$f(x) = \begin{cases} 1 & \text{if } x = 2 \\ 0 & \text{otherwise} \end{cases}$$

Hint: One possible solution uses (a), (b) and the fact that $x = 2$ is equivalent to $2 \leq x \wedge x \leq 2$.

Question 3 $(3 + 0 = 3$ points)

Let $N = (S, T, F)$ be a net of incidence matrix $\mathbf{N}$. Let s, s_1, s_2 be places such that $\mathbf{N}(s, t) = \mathbf{N}(s_1, t) + \mathbf{N}(s_2, t)$ for all $t \in T$, i.e., the row for the place s is the sum of the rows for the places s_1 and s_2 in the matrix $\mathbf{N}$.

(a) A marking M is called *good* if $M(s) = M(s_1) + M(s_2)$. Show that if M is a good marking and M' is reachable from M, then M' is also a good marking. **Hint:** Find a suitable place invariant.

(b) For any good marking M, show that a marking D is a deadlock of (N, M) if and only if marking D_s is a deadlock of (N_s, M_s). Here, N_s is the subnet obtained from N by removing the place s together with its input and output arcs, M_s is the marking of N_s satisfying $M_s(s') = M(s')$ for every $s' \in S \setminus \{s\}$, and similarly D_s is the marking of N_s satisfying $D_s(s') = D(s')$ for every $s' \in S \setminus \{s\}$.
Edit: This question, as currently stated, is wrong because of transitions that remove a token and put back a token in the same place. A counterexample for this claim, along with a proof of the modified version of the claim is given in the solutions. For this reason, we have removed the 4 points associated to this subpart, reduced the total points to 36 and adjusted the points for passing accordingly. **Since one direction of this claim is still true, students who have given partial answers to this question have been given bonus points.**

Question 4 $(4 + 8 = 12$ points$)$

A place s of a Petri net (N, M_0) with $N = (S, T, F)$ is said to be *floodable* if the following is true: For every number k, there is a marking M and an occurrence sequence σ of **length at most** $k \cdot |T|$ such that $M_0 \xrightarrow{\sigma} M$ and $M(s) \geq k$.

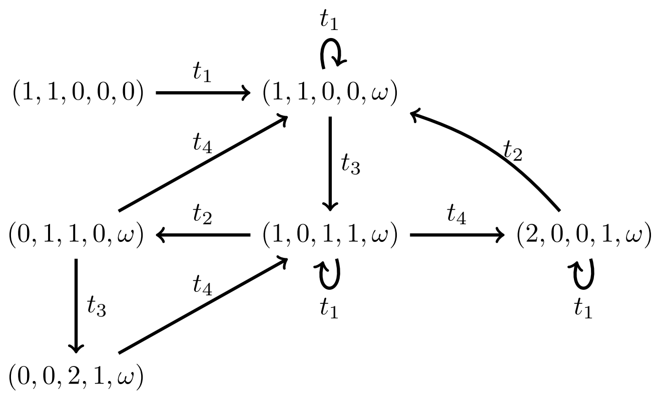
Note that if a place is floodable then it is unbounded. But the converse is in general **not true**.

(a) Give an example of an acyclic net (N, M_0) with at most 5 places and 5 transitions and a place s of N such that s is unbounded, but s is not floodable.

(b) ★ Suppose (N, M_0) is a live T-system such that there is **exactly one place** s which is unbounded and all other places are 1-bounded. Prove that s is floodable.

Solution 1 $(3+3+6=12$ points)

(a) The following is the coverability graph of (N, M_0).



(b) The following is a run leading to a dead marking.

$$(1,1,0,0,0) \xrightarrow{t_1} (1,1,0,0,1) \xrightarrow{t_3} (1,0,1,1,0) \xrightarrow{t_2} (0,1,1,0,1) \xrightarrow{t_3} (0,0,2,1,0)$$

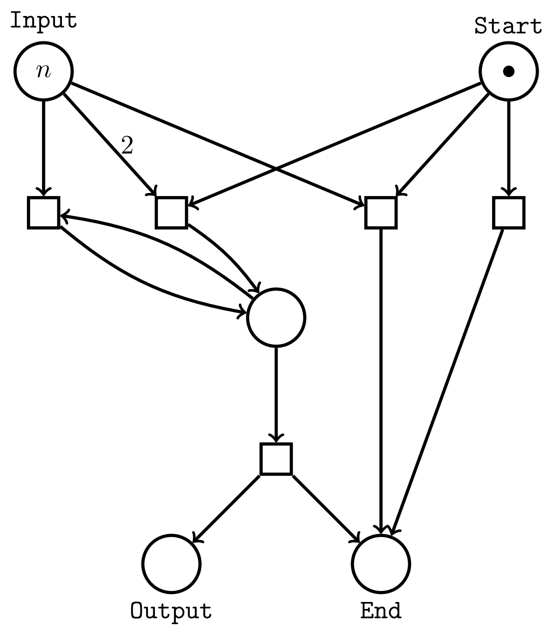
(c) Suppose $M(s_5) > 0$. Since $M(s_1) > 0$ and $M(s_4) > 0$, it follows that we can fire the sequence t_2, t_3, t_4 from M to reach a marking M' where M' is exactly the same as M, except that $M'(s_5) = M(s_5) - 1$. If $M'(s_5) > 0$, we can fire the sequence t_2, t_3, t_4 again from M' to reduce the number of tokens in s_5 by one. By repeating this procedure, from M we can reach a marking M_1 which is the same as M except that $M_1(s_5) = 0$.

Since $M_1(s_1) > 0$ and $M_1(s_4) > 0$, it follows that we can fire the sequence t_2, t_3 from M_1 to reach a marking M_1' which is the same as M_1 except that $M_1'(s_1) = M_1(s_1) - 1$ and $M_1'(s_3) = M_1(s_3) + 1$. If $M_1'(s_1) > 0$, we can fire t_2, t_3 again to reduce the number of tokens in s_1 by one. By repeating this procedure, from M_1 we can reach a marking M_2 such that $M_2(s_1) = M_2(s_5) = 0$.

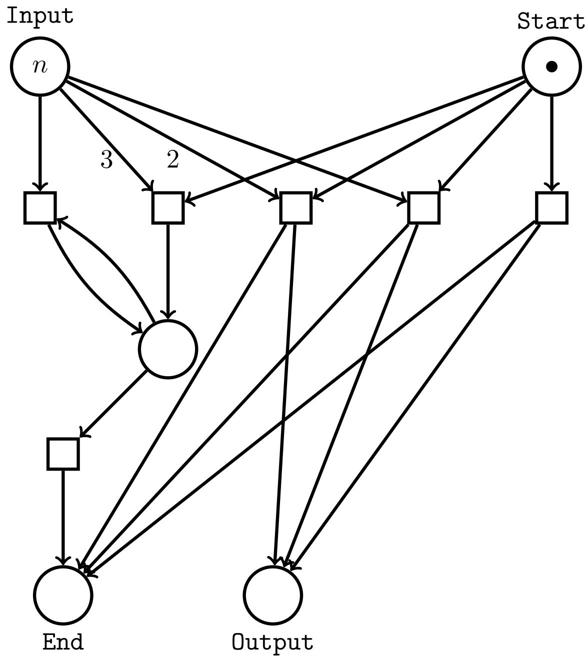
Since $\{s_1, s_5\}$ is a siphon and since $M_2(s_1) = M_2(s_5) = 0$, it follows that if L is any marking reachable from M_2, then $L(s_1) = L(s_5) = 0$. Hence, it follows that the transition t_1 can never be fired from any reachable marking of M_2 and so M_2 is not live. Consequently, this also proves that M is not live.

Solution 2 $(3+3+3=9$ points)

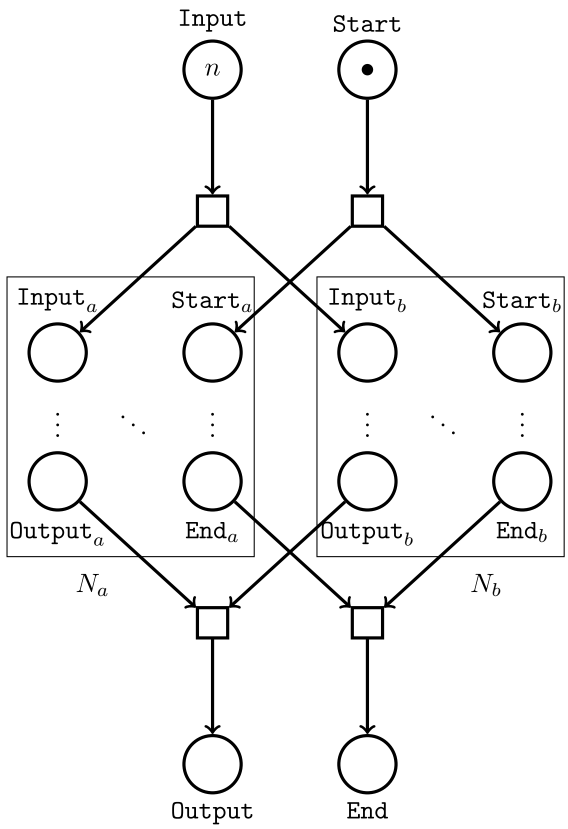
(a) The following is one possible solution.



(b) The following is one possible solution.



(c) The following solution uses the solutions for a) and b). Here, N_a and N_b are the nets for the subparts a) and b) respectively. The places $\texttt{Start}_a, \texttt{Input}_a, \texttt{End}_a$ and $\texttt{Output}_a$ denote the start, input, end and output places of N_a respectively. Similarly for b).



Solution 3 $\quad (3+0=3$ points)

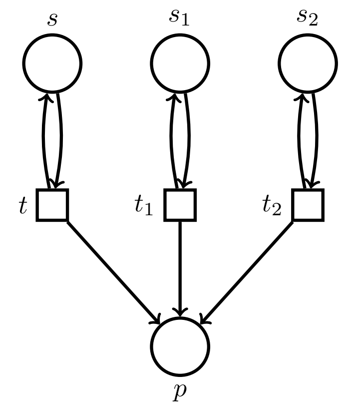
(a) Let $I : S \to \mathbb{N}$ be the vector given by $I(s) = 1, I(s_1) = -1, I(s_2) = -1$ and $I(s') = 0$ for all other places s'. Since $\mathbf{N}(s,t) = \mathbf{N}(s_1,t) + \mathbf{N}(s_2,t)$ for every transition t, it follows that $I \cdot \mathbf{N} = 0$ and so I is a S-invariant of N.

Suppose M is a good marking and M' is reachable from M. By the fundamental property of S-invariants, it follows that $I \cdot M = I \cdot M'$. Since M is a good marking, we have that $I \cdot M = 0$ and so it follows that $I \cdot M' = 0$. Since $I \cdot M' = 0$, it means that $M'(s) - M'(s_1) - M'(s_2) = 0$ and so we have that M' is a good marking.

(b) Let D be a reachable marking of (N, M) where M is a good marking.

Suppose D_s is a reachable marking of (N_s, M_s) which is a deadlock. We will now show that D is also a deadlock of (N, M). Let t be any transition of N. Since t is also a transition of N_s, it follows that there is at least one place $s' \in {}^\bullet t$ in N_s such that $D_s(s') = 0$. By definitions of D_s and N_s, it follows that $D(s') = 0$ and $s' \in {}^\bullet t$ in N as well, which proves that D is a deadlock of (N, M).

The other direction is false. For example, let us consider the following net N.



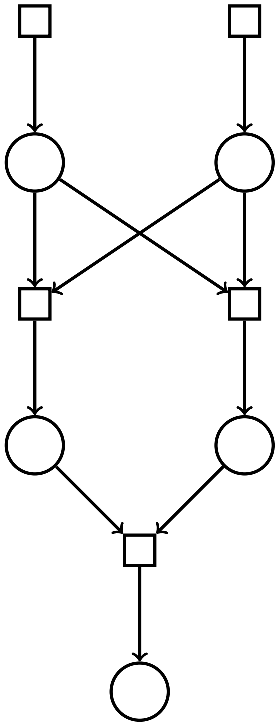
Notice that $\mathbf{N}(s', t) = 0$ for every $s' \in \{s, s_1, s_2\}$ and every transition t and so this net satisfies the property $\mathbf{N}(s,t) = \mathbf{N}(s_1,t) + \mathbf{N}(s_2,t)$ for every transition t. Also notice that the zero marking $\mathbf{0}$ is a deadlock for this net. However, in the net $(N_s, \mathbf{0}_s)$, we can fire the transition t from any marking and so no marking is a deadlock marking.

However, a modified version of this claim is true. Suppose in the net N, there is no transition t such that $s \in {}^\bullet t \cap t^\bullet$. With this assumption, we give a proof of the other direction. Suppose D is a reachable marking of (N, M) such that M is a good marking and D is a deadlock. Let $M \xrightarrow{\sigma} D$. By induction on the length of σ and by using the construction of N_s, M_s and D_s, we can show that $M_s \xrightarrow{\sigma} D_s$ is a valid run in N_s. Let t be any transition of N_s. Since t is also a transition of N, it follows that there is at least one place $s' \in {}^\bullet t$ in N such that $D(s') = 0$. If $s' \neq s$, then $D_s(s') = 0$ in N_s and so D_s cannot fire the transition t.

Suppose $s' = s$. By subpart a), we know that D is a good marking and so $D(s) = D(s_1) + D(s_2)$. Since $D(s) = 0$, it must be the case that $D(s_1) = D(s_2) = 0$. Further, we know that $\mathbf{N}(s,t) = \mathbf{N}(s_1,t) + \mathbf{N}(s_2,t)$ where $\mathbf{N}$ is the incidence matrix of N. Since $s \in {}^\bullet t$, by our assumption $s \notin t^\bullet$ and so $\mathbf{N}(s,t) = -1$. Since there are no weighted arcs, it must be the case that either $\mathbf{N}(s_1,t) = -1$ or $\mathbf{N}(s_2,t) = -1$. Without loss of generality, let $\mathbf{N}(s_1,t) = -1$. Then, $s_1 \in {}^\bullet t$ and $D(s_1) = 0$. This means that $D_s(s_1) = 0$ in N_s and so D_s cannot fire the transition t. Hence, we have shown that D_s cannot fire any transition in N_s and so D_s is a deadlock of (N_s, M_s).

Solution 4 $\quad (4+8=12$ points)

(a) The following is one possible solution.



(b) Let $N = (S, T, F)$ be the given net and let t be the unique transition such that ${}^\bullet s = t$. Let N' be the subnet of N defined as follows:

$$S' = \{s' \in S : \text{ There is a path from } s' \text{ to } t \text{ in } N\}$$

$$T' = \{t' \in T : \text{ There is a path from } t' \text{ to } t \text{ in } N\}$$

$$F' = F \cap ((T' \times S') \cup (S' \times T'))$$

First we show that N' is a T-system. Indeed, suppose s' is a place of N'. In the net N, let ${}^\bullet s' = \{r_1\}$ and $s'^\bullet = \{r_2\}$. We claim that in the net N', ${}^\bullet s' = \{r_1\}$ and $s'^\bullet = \{r_2\}$. Indeed, since s' is a place of N', it follows that there is a path from s' to t in N. This implies that there is also a path from r_1 to t in N and so r_1 is a transition of N'. Since any outgoing path from s' in N has to go through r_2, it also follows that there is a path from r_2 to t in N and so r_2 is a transition of N'. Since F' is the restriction of the flow relation F to S' and T', the require claim follows. Hence, N' is also a T-system.

Let M_0' be the marking of N' given by $M_0'(s) = M_0(s)$ for all $s \in S'$. We now claim that (N', M_0') is live. Indeed, since (N, M_0) is live, by the liveness theorem for T-systems, it follows that $M_0(\gamma) > 0$ for every circuit γ of N. Since every circuit of N' is also a circuit of N and since M_0 and M_0' agree on all $s \in S'$, it follows that $M_0'(\gamma) > 0$ for every circuit γ of N'. By the liveness theorem for T-systems, it follows that (N', M_0') is live.

We now prove that (N', M_0') is 1-bounded. Since, all places of (N, M_0) except s are 1-bounded and since (N', M_0') is a subnet of N where M_0' and M_0 agree on all $s \in S'$, it suffices to show that the place s cannot be present in N'. For the sake of contradiction, suppose s is a place of N'. By definition, there must be a path from s to t in N. Since ${}^\bullet s = \{t\}$, it follows that there is also a path from t to s in N and so s is part of a circuit γ in N. But then the boundedness theorem implies that s must be bounded, which leads to a contradiction. Hence s cannot be present in N' and so (N', M_0') is 1-bounded.

Hence we have shown that (N', M_0') is a live 1-bounded T-system. From the lectures, we know the following fact:

> If (N', M_0') is a live 1-bounded T-system, then there exists a firing sequence σ in which each transition of N' appears exactly once such that $M_0' \xrightarrow{\sigma} M_0'$ in N'.

Applying this fact here we get a firing sequence σ of length $|T'| \leq |T|$ such that $M_0' \xrightarrow{\sigma} M_0'$ in the net N'. This then implies that $M_0 \xrightarrow{\sigma} M_1 \xrightarrow{\sigma} M_2 \dots M_{k-1} \xrightarrow{\sigma} M_k$ is a valid run of length at most $k \cdot |T|$ in the net N where each M_i is exactly the same as M_0 except that $M_i(s') = M_0(s') + i$ for every $s' \in t^\bullet$. In particular, we have that $M_k(s) \geq k$ and so s is floodable.