

## Petri nets — Endterm

- You have **75 minutes** to complete the exam.
- Write your name, Matrikelnummer (immatriculation number) and page number on every sheet.
- Write with a black or blue **pen**. Do not use red or green.
- You can obtain **40 points**, of which 5 are bonus points. You need **15 points** to pass.

**For all the questions of this exam, we only consider Petri nets that are connected, and that do not have weights on the arcs.** The symbol ★ denotes that we consider this question to be a bit harder.

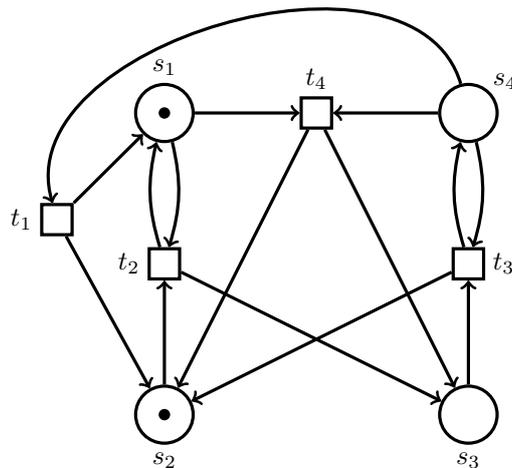
**Question 1 (5 + 3 + 6 = 14 points)**

Let  $N = (S, T, F)$  be a Petri net and let  $\mathbf{0}$  be the marking which has zero tokens in each place of  $N$ , i.e.,  $\mathbf{0}(s) = 0$  for every  $s \in S$ . Let  $\mathcal{G}$  be the coverability graph of  $(N, \mathbf{0})$ .

- (a) Consider an execution of the algorithm which constructs the coverability graph  $\mathcal{G}$ . Assume that at some point along the execution, every node  $M$  of the graph that has already been constructed satisfies the property that for every place  $s$ , either  $M(s) = 0$  or  $M(s) = \omega$ . Show that the next node  $M'$  constructed by the algorithm (if it exists) is also such that for every place  $s$ , either  $M'(s) = 0$  or  $M'(s) = \omega$ .
- (b) Show that if  $L_1$  and  $L_2$  are *reachable markings* of the Petri net  $(N, \mathbf{0})$  (**hence  $L_1$  and  $L_2$  do not have any  $\omega$ -places**), then  $k \cdot L_1 + k \cdot L_2$  is also a reachable marking of the Petri net  $(N, \mathbf{0})$  for every  $k > 0$ .
- (c) ★ Using (b), show that if  $M_1$  and  $M_2$  are  $\omega$ -markings in the coverability graph  $\mathcal{G}$ , then there is an  $\omega$ -marking  $M'$  in the coverability graph  $\mathcal{G}$  such that  $M' \geq M_1 + M_2$ .

**Question 2 (2 + 2 + 1 + 1 = 6 points)**

Consider the following Petri net  $(N, M_0)$ :



- (a) Give a basis for the vector space of **S-invariants** of  $N$ .

- (b) Give a basis for the vector space of **T-invariants** of  $N$ .
- (c) Using only S-invariants and T-invariants of  $N$  can you decide if  $(N, M_0)$  is bounded? If yes, then state whether  $(N, M_0)$  is bounded or not, and justify.
- (d) Using only S-invariants and T-invariants of  $N$  can you decide if  $N$  is well-formed? If yes, then state whether  $N$  is well-formed or not, and justify.

**Question 3 (2 + 2 + 2 = 6 points)**

Consider the same Petri net  $(N, M_0)$  from Question 2.

- (a) Let  $A_1 = \{s_1, s_4\}$ ,  $A_2 = \{s_3, s_4\}$ ,  $A_3 = \{s_1, s_2\}$ ,  $A_4 = \{s_1, s_3, s_4\}$ . State if each  $A \in \{A_1, A_2, A_3, A_4\}$  is a **siphon** or not, and justify.
- (b) Let  $B_1 = \{s_2, s_4\}$ ,  $B_2 = \{s_2, s_3\}$ ,  $B_3 = \{s_1, s_3, s_4\}$ ,  $B_4 = \{s_4\}$ . State if each  $B \in \{B_1, B_2, B_3, B_4\}$  is a **trap** or not, and justify.
- (c) Let  $M$  be the marking such that  $M(s_1) = 1$  and  $M(s) = 0$  if  $s \neq s_1$ . Using only the siphons and traps from (a) and (b), can you decide if  $M_0$  can reach  $M$ ? If yes, then state whether  $M_0$  can reach  $M$ , and justify.

**Question 4 (3 + 5 = 8 points)**

- (a) A net  $N$  is called *bad* if there is a marking  $M$  for which  $(N, M)$  is live and 2-bounded, but there is no marking  $M'$  for which  $(N, M')$  is live and 1-bounded. Give an example of a bad net.
- (b) ★ **(Bonus question)** Prove that no free-choice net can be bad. (**Hint:** Use Commoner's Liveness theorem and Hack's boundedness theorem).

**Question 5 (2 + 4 = 6 points)**

- (a) Give an example of a 1-bounded Petri net  $(N, M_0)$  with exactly 4 places such that the number of reachable markings of  $(N, M_0)$  is 16.
- (b) Prove that if  $(N, M_0)$  is a 1-bounded, **cyclic** Petri net with exactly 4 places, then the number of reachable markings of  $(N, M_0)$  is *strictly less than* 15. (**Hint:** Use the Monotonicity Lemma).

**Solution 1 (5 + 3 + 6 = 14 points)**

- (a) To construct the next node, the algorithm first picks an  $\omega$ -marking  $M$  already in the constructed graph and a transition  $t \in \text{enabled}(M)$  and computes  $M' = \text{fire}(M, t)$ . Then it runs the procedure  $\text{AddOmegas}(M, M', V, E)$ , where we compare the current  $M'$  we have with all  $\omega$ -markings  $M''$  which can reach  $M'$  along some path in the graph, and if  $M'' < M'$ , then we update  $M'$  as  $M' := M' + (M' - M'') \cdot \omega$ . Finally, if the resulting  $M'$  is not already a node in the graph, it adds  $M'$  to the graph.

Suppose the  $\text{AddOmegas}$  procedure updates  $M'$  at some point by replacing  $M'$  with  $M' + (M' - M'') \cdot \omega$  for some  $\omega$ -marking  $M''$  already in the graph. We claim that  $\bar{M} = M' + (M' - M'') \cdot \omega$  satisfies the property that  $\bar{M}(s) = 0$  or  $\bar{M}(s) = \omega$  for every place  $s$ . Let  $s$  be some place. If  $M'(s) = \omega$ , then  $\bar{M}(s)$  is also  $\omega$  and hence we are done. Suppose  $M'(s) \neq \omega$ . The  $\text{AddOmegas}$  procedure performs an update only if  $M'' < M'$ . By assumption,  $M''$  satisfies the property that  $M''(p) = 0$  or  $M''(p) = \omega$  for every place  $p$ . These two facts together imply that  $M''(s) = 0$  and so  $\bar{M}(s) = M'(s) + M'(s) \cdot \omega$ , which is always either 0 or  $\omega$ . Hence, we are done.

Hence, all that is required to prove is that the  $\text{AddOmegas}$  procedure performs an update to  $M'$  at least once. But this will always happen as  $\mathbf{0}$  is an  $\omega$ -marking which can reach any marking in the graph and is always strictly less than every other marking. Hence,  $\text{AddOmegas}$  updates  $M'$  at least once and so the final  $M'$  that we add to the graph satisfies the property that  $M'(s) = 0$  or  $M'(s) = \omega$  for every place  $s$ .

- (b) By the monotonicity lemma, we know that if  $\mathbf{0} \xrightarrow{\sigma_1} L_1$  and  $\mathbf{0} \xrightarrow{\sigma_2} L_2$ , then  $\mathbf{0} \xrightarrow{\sigma_1} L_1 = \mathbf{0} + L_1 \xrightarrow{\sigma_1} 2 \cdot L_1 = \mathbf{0} + 2 \cdot L_1 \dots \xrightarrow{\sigma_1} k \cdot L_1$ . Again by the monotonicity lemma we have  $k \cdot L_1 = \mathbf{0} + k \cdot L_1 \xrightarrow{\sigma_2} k \cdot L_1 + L_2 = \mathbf{0} + k \cdot L_1 + L_2 \dots \xrightarrow{\sigma_2} k \cdot L_1 + k \cdot L_2$ . Hence,  $\mathbf{0}$  can reach  $k \cdot L_1 + k \cdot L_2$ .
- (c) Suppose  $M_1$  and  $M_2$  are  $\omega$ -markings in the coverability graph. By Lemma 3.2.7 of the lecture notes, there exists a reachable marking  $L_1$  such that  $L_1(s) = M_1(s)$  for every normal place  $s$  of  $M_1$  and  $L_1(s) > 1$  for every  $\omega$ -place of  $s$ . Similarly, there exists a marking  $L_2$  for  $M_2$ . By b),  $L_1 + L_2$  is also reachable. By the property of the coverability graph, there must be an  $\omega$ -marking  $M'$  such that  $M' \geq L_1 + L_2$ . We claim that  $M' \geq M_1 + M_2$  as well.

If  $(M_1 + M_2)(s) = 0$  for some place  $s$ , then  $(L_1 + L_2)(s) = 0$  and so  $M'(s) \geq (M_1 + M_2)(s)$ . Suppose  $(M_1 + M_2)(s) > 0$  for some place  $s$ . Then  $(L_1 + L_2)(s) > 0$ . Since  $M' \geq L_1 + L_2$ , by a), we get that  $M'(s) = \omega$ . Hence,  $M'(s) \geq (M_1 + M_2)(s)$  and so we are done.

**Solution 2 (2 + 2 + 1 + 1 = 6 points)**

- (a) Recall that  $I$  is an  $S$ -invariant if and only if  $\sum_{s \in \bullet t} I(s) = \sum_{s \in t \bullet} I(s)$  for every  $t \in T$ . This gives rise to the following system of equations:

$$\begin{aligned} I(s_4) &= I(s_1) + I(s_2), \\ I(s_1) + I(s_2) &= I(s_1) + I(s_3), \\ I(s_4) + I(s_3) &= I(s_4) + I(s_2), \\ I(s_1) + I(s_4) &= I(s_2) + I(s_3) \end{aligned}$$

which is equivalent to:

$$\begin{aligned} I(s_2) &= I(s_3) \text{ (line 2),} \\ 2 \cdot I(s_1) + I(s_2) &= 2 \cdot I(s_2) \text{ (lines 1 plus 4), i.e.} \\ 2 \cdot I(s_1) &= I(s_2) = I(s_3), \text{ and} \\ 3 \cdot I(s_1) &= I(s_4) \text{ (line 1)} \end{aligned}$$

Therefore, each  $S$ -invariant  $I$  is fully determined by  $I(s_1)$ , and the vector space of  $S$ -invariants is given by:

$$x \cdot (1 \quad 2 \quad 2 \quad 3) \quad \text{for } x \in \mathbb{Q}.$$

- (b) Recall that  $J$  is an  $T$ -invariant if and only if  $\sum_{t \in \bullet s} J(s) = \sum_{t \in s \bullet} J(t)$  for every  $s \in S$ . This gives rise to the following system of equations:

$$\begin{aligned} J(t_1) + J(t_2) &= J(t_1) + J(t_4), \\ J(t_1) + J(t_3) + J(t_4) &= J(t_2), \\ J(t_2) + J(t_4) &= J(t_3), \\ J(t_3) &= J(t_3) + J(t_1) + J(t_4) \end{aligned}$$

which is equivalent to:

$$\begin{aligned} J(t_1) &= J(t_4), \text{ and} \\ 0 &= J(t_1) + J(t_4), \text{ so} \\ 0 &= J(t_1) = J(t_4), \text{ and} \\ J(t_3) &= J(t_2) \end{aligned}$$

Therefore, each  $T$ -invariant  $J$  is fully determined by  $J(t_2)$ , and the vector space of  $T$ -invariants is given by:

$$x \cdot (0 \quad 1 \quad 1 \quad 0) \quad \text{for } x \in \mathbb{Q}.$$

- (c) If a Petri net has a positive  $S$ -invariant, then it is bounded from any initial marking. By (a), taking  $x > 0$  yields a positive  $S$ -invariant; for example  $(1 \quad 2 \quad 2 \quad 3)$ . Therefore, yes we can decide, and  $(N, M_0)$  is bounded.
- (d) Every well-formed net has a positive  $T$ -invariant. But from (b), we know that every  $T$ -invariant is equal to 0 in the first and last component, so it cannot be positive. Therefore, yes we can decide, and  $N$  is not well-formed.

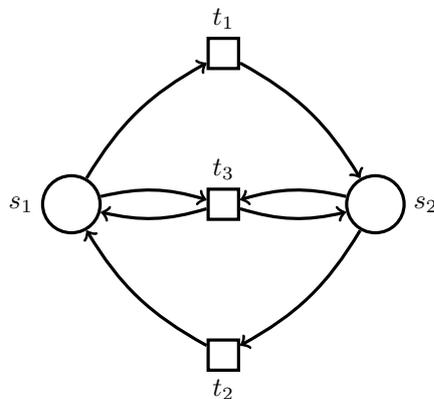
**Solution 3 (2 + 2 + 2 = 6 points)**

Let  $P$  be the set of places of  $N$  and  $T$  the set of transitions.

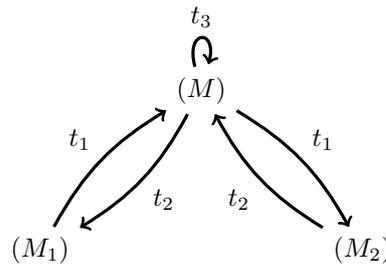
- (a) A set  $R \subseteq P$  is a siphon iff  $\bullet R \subseteq R^\bullet$ . This is the case if for all transitions  $t \in T$ , if for some  $p \in t^\bullet$  we have  $p \in R$ , then for some  $p' \in \bullet t$  we have  $p' \in R$ .
1.  $A_1$  is a siphon:  $\bullet A_1 = \{t_1, t_2, t_3\} \subseteq A_1^\bullet = T$
  2.  $A_2$  is not a siphon:  $t_2 \in \bullet A_2$  but  $t_2 \notin A_2^\bullet$
  3.  $A_3$  is not a siphon:  $t_1 \in \bullet A_3$  but  $t_1 \notin A_3^\bullet$
  4.  $A_4$  is a siphon:  $\bullet A_4 = T \subseteq A_4^\bullet = T$
- (b) A set  $R \subseteq P$  is a trap iff  $R^\bullet \subseteq \bullet R$ .
1.  $B_1$  is not a trap:  $t_2 \in B_1^\bullet$  but  $t_2 \notin \bullet B_1$
  2.  $B_2$  is a trap:  $B_2^\bullet = \{t_2, t_3\} \subseteq \bullet B_2 = T$
  3.  $B_3 = A_4$  is a trap:  $B_3^\bullet = T \subseteq \bullet B_3 = T$
  4.  $B_4$  is not a trap:  $t_4 \in B_4^\bullet$  but  $t_4 \notin \bullet B_4$
- (c) Marked traps stay marked: if  $M_0(R) > 0$  for some trap  $R$ , then  $M(R) > 0$  for all reachable marking  $M$ .  $B_2$  is a trap marked at  $M_0$  but not at  $M$ . Therefore we can decide that  $M_0$  cannot reach  $M$ .

**Solution 4 (3 + 5 = 8 points)**

- (a) Consider the following Petri net  $N$ :



If we put 1 token each in  $s_1$  and  $s_2$ , then the resulting marking  $M$  is such that  $(N, M)$  is live and 2-bounded. Indeed, if  $M_1$  is the marking given by  $M_1(s_1) = 2$  and  $M_1(s_2) = 0$  and  $M_2$  is the marking given by  $M_2(s_1) = 0$  and  $M_2(s_2) = 2$ , then the reachability graph of  $(N, M)$  is given by the following, from which we can infer that  $(N, M)$  is live and 2-bounded.



Suppose there exists a marking  $M'$  such that  $(N, M')$  is live and 1-bounded. Since  $(N, M')$  is live, it cannot be the case that  $M'(s_1) = 0$  and  $M'(s_2) = 0$ . Further, since  $(N, M')$  is 1-bounded, it cannot be the case that  $M'(s_1) > 0$  and  $M'(s_2) > 0$ , as otherwise  $M'$  can reach a marking with at least 2 tokens in  $s_1$ . Hence, either  $M'(s_1) > 0$  and  $M'(s_2) = 0$  or  $M'(s_1) = 0$  and  $M'(s_2) > 0$ . Let us consider the first case. (The second case is similar).

Since  $(N, M')$  is 1-bounded,  $M'(s_1) = 1$ . Notice that none of the transitions can create or destroy tokens. This means that if  $M''$  is any reachable marking of  $M'$ , then  $M''$  has exactly one token. But this also implies that the transition  $t_3$  can never be fired, leading to a contradiction that  $(N, M')$  is live.

**Remark:** Explanation is not required and full points have been awarded to any correct solution.

- (b) Suppose  $N$  is a free-choice net for which there exists a marking  $M$  such that  $(N, M)$  is live and 2-bounded, but not 1-bounded.

We now construct another live marking  $L$  of  $N$  such that

- (1)  $L(S') \leq M(S')$  for every S-component of  $N$ , and
- (2)  $L(S') < M(S')$  for some S-component of  $N$ .

By Proposition 5.3.15 (Place bounds) in the lecture notes, this means that at least one place of  $N$  has a smaller bound under  $L$  as under  $M$ . Iterating this procedure, will then yield a live and 1-bounded marking of  $N$ .

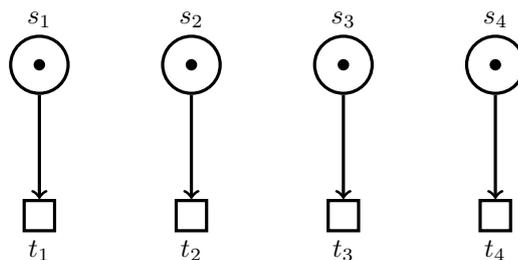
Since  $(N, M)$  is live and 2-bounded, but not 1-bounded, there exists a reachable marking  $M'$  of  $(N, M)$  and a place  $s$  such that  $M'(s) = 2$ . Let  $L$  be the marking which puts exactly one token in  $s$  and as many tokens as  $M$  elsewhere.

Since  $(N, M)$  is live, by Commoner's liveness theorem, every proper siphon of  $N$  has a trap marked at  $M$ . It is then clear that by Commoner's liveness theorem,  $(N, L)$  is also live.

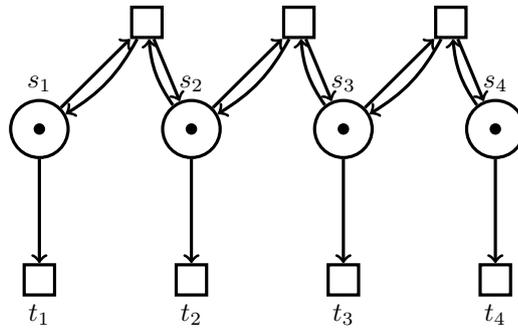
We now show that  $L$  satisfies conditions (1) and (2). Condition (1) simply follows by the definition of  $L$ . For condition (2), first note that since  $(N, M)$  is bounded, by Hack's boundedness theorem, every place of  $N$  belongs to an  $S$ -component. Hence, the place  $s$  also belongs to an  $S$ -component  $S'$ . By definition of  $L$ , we have that  $L(S') < M(S')$ .

**Solution 5 (2 + 4 = 6 points)**

- (a) The following Petri net is such an example:



Or a connected version:



**Remark:** Full points have been awarded to any correct solution (even non-connected).

- (b) Suppose  $(N, M_0)$  is a 1-bounded, cyclic Petri net with exactly 4 places. First, notice that if  $M_0$  can reach a marking  $M$  such that  $M > M_0$ , then by the Monotonicity lemma,  $(N, M_0)$  cannot be 1-bounded. Similarly, if  $M_0$  can reach a marking  $M$  such that  $M < M_0$ , then because  $(N, M_0)$  is cyclic,  $M$  can reach  $M_0$  and so by the previous argument,  $(N, M)$  is not 1-bounded and so  $(N, M_0)$  is also not 1-bounded. Hence, if  $M_0$  can reach  $M$  then  $M_0$  and  $M$  are incomparable. This means that  $M_0$  cannot reach the marking which puts zero tokens in every place of  $N$ . Further, since  $(N, M_0)$  is 1-bounded, this also means that  $M_0$  cannot reach the marking which puts one token in every place of  $N$ . Since,  $M_0$  is 1-bounded, this means that the number of reachable markings of  $(N, M_0)$  is strictly less than 15.