First-order Predicate Logic
Theories

Preliminary Definitions

Definition

A signature Σ is a set of predicate and function symbols.

A Σ-formula is a formula that contains only predicate and function symbols from Σ.

A Σ-structure is a structure that interprets all predicate and function symbols from Σ.

Definition

A Σ-sentence is a closed Σ-formula.

Convention: we assume that a signature Σ has been fixed, and drop Σ in Σ-formula, Σ-structure, or Σ-sentence. That is, we silently assume that all formulas, structures, and sentences are over the same fixed signature Σ.

Theories

Definition

A theory is a (finite or infinite) set of sentences S closed under consequence: If $S \models F$ and F is a sentence, then $F \in S$.

Fact

The set A of all sentences is a theory: If $S \models F$ for a sentence F, then in particular F is a sentence and so $F \in S$.

The set V of all valid sentences is a theory: If $V \models F$ for a sentence F, then F is valid and so $F \in V$.

$V \subseteq S$ hods for every theory S: If F is a valid sentence, then $\models F$. It follows $S \models F$ and, since S is a theory, $F \in S$.

There are two ways to define interesting theories:

- As the set of sentences satisfied by a fixed structure.
- As the set of consequences of a fixed set of sentences.

Theories from structures

Definition

Given a structure $\mathcal{A}$, let $Th(\mathcal{A})$ denote the set of all sentences true in $\mathcal{A}$. That is, $Th(\mathcal{A}) := \{F \mid F \text{ is a sentence and } \mathcal{A} \models F\}$.

Lemma

For every structure $\mathcal{A}$ and sentence F: $\mathcal{A} \models F$ iff $Th(\mathcal{A}) \models F$.

Proof.

"$\Rightarrow$": $\mathcal{A} \models F \Rightarrow F \in Th(\mathcal{A}) \Rightarrow Th(\mathcal{A}) \models F$.

"$\Leftarrow$": Assume $Th(\mathcal{A}) \models F$. We prove $\mathcal{A} \models Th(\mathcal{A})$, which, together with $\mathcal{A} \models F$, implies $\mathcal{A} \models F$. To prove $\mathcal{A} \models Th(\mathcal{A})$, let $G \in Th(\mathcal{A})$. We have $\mathcal{A} \models G$ by definition of $Th(\mathcal{A})$.

Corollary

$Th(\mathcal{A})$ is a theory.

Proof. Assume $Th(\mathcal{A}) \models F$. By the lemma above, $\mathcal{A} \models F$. From the definition of $Th(\mathcal{A})$ we get $F \in Th(\mathcal{A})$.

Example

Notation: $(\mathbb{Z}, +, \leq)$ denotes the structure with universe $\mathbb{Z}$ and the standard interpretations for the symbols $+$ and $\leq$.

The same notation is used for other standard structures where the interpretation of a symbol is clear from the symbol.

Example (Linear integer arithmetic)

$\mathit{Th}(\mathbb{Z}, +, \leq)$ is the set of all sentences over the signature $\{+, \leq\}$ that are true in the structure $(\mathbb{Z}, +, \leq)$.

Famous numerical theories

$Th(\mathbb{R}, +, <, =)$ is called linear real arithmetic.

It is decidable.

$Th(\mathbb{R}, +, *, <, =)$ is called real arithmetic.

It is decidable.

$Th(\mathbb{Z}, +, <, =)$ is called linear integer arithmetic or Presburger arithmetic.

It is decidable.

$Th(\mathbb{Z}, +, *, <, =)$ is called integer arithmetic.

It is not even semidecidable (= r.e.).

Decidability via special algorithms.

Theories from axioms

Definition

Let S be a set of sentences. $Cn(S)$ denotes the set of consequences of S: $Cn(S) = \{F \mid F \text{ is a sentence and } S \models F\}$

Examples

$Cn(\emptyset)$ is the set of valid sentences.

$Cn(\{\forall x \forall y \forall z\ (x * y) * z = x * (y * z)\})$ is the set of sentences that are true in all semigroups.

Lemma

For every set S of sentences, the set $Cn(S)$ is a theory.

Proof. We show: if F is a sentence and $Cn(S) \models F$ then $F \in Cn(S)$.

By the def. of $Cn(S)$ we have $S \models Cn(S)$. By assumption $Cn(S) \models F$, and so $S \models F$ by transitivity of $\models$. From the definition of $Cn(S)$ we get $F \in Cn(S)$.

Axioms

Definition

Let S be a set of sentences.

A theory T is axiomatized by S if $T = Cn(S)$.

A theory T is axiomatizable if there is some decidable or recursively enumerable S that axiomatizes T.

A theory T is finitely axiomatizable if there is some finite S that axiomatizes T.

Completeness and elementary equivalence

Definition

A theory T is complete if for every sentence F, $T \models F$ or $T \models \neg F$.

Fact

$Th(\mathcal{A})$ is complete for every structure $\mathcal{A}$.

$Cn(\{\forall x \forall y \forall z\ (x * y) * z = x * (y * z)\})$ is incomplete: neither $\forall x \forall y\ x * y = y * x$ nor its negation belong to it.

Definition

Two structures $\mathcal{A}$ and $\mathcal{B}$ are elementarily equivalent if $Th(\mathcal{A}) = Th(\mathcal{B})$.

Theorem

A theory T is complete iff all its models are elementarily equivalent.

Theorem

A theory T is complete iff all its models are elementarily equivalent.

Proof. We prove that T is incomplete iff two of its models are not elementarily equivalent.

"$\Rightarrow$:" Assume T is incomplete. Then $T \not\models \neg F$ and $T \not\models F$ for some F. So $\mathcal{A}(\neg F) = 0$ (thus $\mathcal{A}(F) = 1$) for some model $\mathcal{A}$ of T, and $\mathcal{B}(F) = 0$ for some model $\mathcal{B}$ of T.

It follows $F \in Th(\mathcal{A}) \setminus Th(\mathcal{B})$, and so $Th(\mathcal{A}) \neq Th(\mathcal{B})$.

"$\Leftarrow$:" Assume two models $\mathcal{A}$ and $\mathcal{B}$ of T are not elementarily equivalent. W.l.o.g. $Th(\mathcal{A}) \setminus Th(\mathcal{B}) \neq \emptyset$ and so $\mathcal{A}(F) = 1$ and $\mathcal{B}(F) = 0$ for some F.

We prove $T \not\models F$ and $T \not\models \neg F$. If $T \models F$, then, since $\mathcal{B} \models T$ we have $\mathcal{B} \models F$, contradicting $\mathcal{B}(F) = 0$. If $T \models \neg F$ then, since $\mathcal{A} \models T$, we have $\mathcal{A} \models \neg F$, contradicting $\mathcal{A}(F) = 1$.