First-Order Logic
Basic Proof Theory

Gebundene Namen sind Schall und Rauch

We permit ourselves to identify formulas that differ only in the names of bound variables.

Example

$\forall x \exists y \, P(x, y) \; = \; \forall u \exists v \, P(u, v)$

Recall: renaming must not capture free variables!

$\forall x \, P(x, y) \; \neq \; \forall y \, P(y, y)$

In the folllowing: Substitution $F[t/x]$ assumes that bound variables in F are automatically renamed to avoid capturing free variables.

Example

$(\exists x \, P(x, y))[x/y] \; = \exists x' \, P(x', x)$

All proof systems below are extensions
of the corresponding propositional systems

Sequent Calculus

Recall: Sequent Calculus rules

$$\frac{}{\bot, \Gamma \Rightarrow \Delta} \; \bot L \qquad \frac{}{A, \Gamma \Rightarrow A, \Delta} \; Ax$$

$$\frac{\Gamma \Rightarrow F, \Delta}{\neg F, \Gamma \Rightarrow \Delta} \; \neg L \qquad \frac{F, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \neg F, \Delta} \; \neg R$$

$$\frac{F, G, \Gamma \Rightarrow \Delta}{F \wedge G, \Gamma \Rightarrow \Delta} \; \wedge L \qquad \frac{\Gamma \Rightarrow F, \Delta \quad \Gamma \Rightarrow G, \Delta}{\Gamma \Rightarrow F \wedge G, \Delta} \; \wedge R$$

$$\frac{F, \Gamma \Rightarrow \Delta \quad G, \Gamma \Rightarrow \Delta}{F \vee G, \Gamma \Rightarrow \Delta} \; \vee L \qquad \frac{\Gamma \Rightarrow F, G, \Delta}{\Gamma \Rightarrow F \vee G, \Delta} \; \vee R$$

$$\frac{\Gamma \Rightarrow F, \Delta \quad G, \Gamma \Rightarrow \Delta}{F \rightarrow G, \Gamma \Rightarrow \Delta} \; \rightarrow L \qquad \frac{F, \Gamma \Rightarrow G, \Delta}{\Gamma \Rightarrow F \rightarrow G, \Delta} \; \rightarrow R$$

Rules for quantifiers

We add the following rules:

$$\frac{F[t/x], \forall x\, F, \Gamma \Rightarrow \Delta}{\forall x\, F, \Gamma \Rightarrow \Delta}\ \forall L \qquad \frac{\Gamma \Rightarrow F[y/x], \Delta}{\Gamma \Rightarrow \forall x\, F, \Delta}\ \forall R\ (*)$$

$$\frac{F[y/x], \Gamma \Rightarrow \Delta}{\exists x\, F, \Gamma \Rightarrow \Delta}\ \exists L\ (*) \qquad \frac{\Gamma \Rightarrow F[t/x], \exists x\, F, \Delta}{\Gamma \Rightarrow \exists x\, F, \Delta}\ \exists R$$

$(*)$: y not free in the conclusion of the rule

Note: $\forall L$ and $\exists R$ do not delete the principal formula, and so termination no longer guaranteed.

Soundness

Lemma
For every quantifier rule $\frac{S'}{S}$, $|S|$ *and* $|S'|$ *are equivalid.*

Theorem (Soundness)
If $\vdash_G S$ *then* $\models |S|$.

Proof induction on the size of the proof of $\vdash_G S$ using the above lemma and the corresponding propositional lemma (stating $|S| \equiv |S_1| \land \ldots \land |S_n|$ for every rule $\dfrac{S_1 \quad \cdots \quad S_n}{S}$).

Completeness Proof

Construct counter model
from (possibly infinite!) failed proof search.

Let $e_0, e_1, \ldots$ be an enumeration of all terms
(over some given set of function symbols and variables)

Proof search

Construct proof tree incrementally:

1. Pick some unproved leaf $\Gamma \Rightarrow \Delta$ such that some rule is applicable.
2. Pick some principal formula in $\Gamma \Rightarrow \Delta$ fairly and apply rule.

   $\forall R$, $\exists L$: pick some arbitrary new y.

   $\forall L$, $\exists R$:

$$t = \begin{cases} e_0 & \text{if the p.f. has never been instantiated} \\ & \text{(on the path to the root)} \\ e_{i+1} & \text{if the previous instantiation of the p.f.} \\ & \text{(on the path to the root) used } e_i \end{cases}$$

Failed proof search: there is a branch A such that either A ends in a sequent where no rule is applicable or A is infinite.

Construction of Herbrand countermodel $\mathcal{A}$ from A

Define a structure $\mathcal{A}$ by:

$U^{\mathcal{A}}$ $=$ all terms over the function symbols and variables in A.

$f^{\mathcal{A}}(t_1, \ldots, t_n)$ $= f(t_1, \ldots, t_n)$.

$P^{\mathcal{A}}$ $= \{(t_1, \ldots, t_n) \mid P(t_1, \ldots, t_n) \in \Gamma$ for some $\Gamma \Rightarrow \Delta \in A\}$.

Theorem

For all $\Gamma \Rightarrow \Delta \in A$*, for all* $F \in \Gamma \cup \Delta$ *:* $\mathcal{A}(F) = \begin{cases} 1 & \text{if } F \in \Gamma \\ 0 & \text{if } F \in \Delta \end{cases}$

In particular, $\Gamma_1 \cap \Delta_2$ *for any two sequents* $\Gamma_1 \Rightarrow \Delta_1$ *and* $\Gamma_2 \Rightarrow \Delta_2$ *of* A*.*

Proof by induction on the structure of F.

Basis: $F = P(t_1, \ldots, t_n)$.

Assume $F \in \Gamma$. Then $\mathcal{A}(F) = 1$ by definition.

Assume $F \in \Delta$. Then F does not belong to any Γ of A; otherwise A would end with an application of Ax. So $\mathcal{A}(F) = 0$.

$$\frac{}{F, \Gamma \Rightarrow F, \Delta}\ Ax$$

where F atomic.

Step: F is not atomic . Then F is the principal formula of some sequent $\Gamma \Rightarrow \Delta \in A$ (fairness!).

We consider several cases, depending on the form of F and whether $F \in \Gamma$ or $F \in \Delta$:

$F = \neg G$:

Take any step $\dfrac{\tilde{\Gamma} \Rightarrow \tilde{\Delta}}{\Gamma \Rightarrow \Delta}$ of A.

If $\neg G \in \Gamma$ then $G \in \tilde{\Delta}$.
By IH $\mathcal{A}(G) = 0$ and so $\mathcal{A}(\neg G) = 1$.

If $\neg G \in \Delta$ then $G \in \tilde{\Gamma}$.
By IH $\mathcal{A}(G) = 1$ and so $\mathcal{A}(\neg G) = 0$.

$$\frac{\Gamma' \Rightarrow G, \Delta}{\neg G, \Gamma' \Rightarrow \Delta} \neg L$$

$$\frac{G, \Gamma \Rightarrow \Delta'}{\Gamma \Rightarrow \neg G, \Delta'} \neg R$$

$F = G_1 \wedge G_2$:

Take any $\dfrac{\tilde{\Gamma} \Rightarrow \tilde{\Delta}}{\Gamma \Rightarrow \Delta}$ of A.

If $G_1 \wedge G_2 \in \Gamma$ then $G_1 \in \tilde{\Gamma}$ and $G_2 \in \tilde{\Gamma}$.
By IH $A(G_1) = \mathcal{A}(G_2) = 1$, and so
$\mathcal{A}(G_1 \wedge G_2) = 1$

$$\frac{G_1, G_2, \Gamma \Rightarrow \Delta}{G_1 \wedge G_2, \Gamma' \Rightarrow \Delta} \wedge L$$

$$\frac{\Gamma \Rightarrow G_1, \Delta \quad \Gamma \Rightarrow G_2, \Delta}{\Gamma \Rightarrow G_1 \wedge G_2, \Delta} \wedge R$$

Completeness

Corollary

If proof search with root $\Gamma \Rightarrow \Delta$ *fails, then there is a structure* $\mathcal{A}$ *such that* $\mathcal{A}(\bigwedge \Gamma \rightarrow \bigvee \Delta) = 0$.

Example

$\exists x\, P(x) \Rightarrow \forall x\, P(x)$

Corollary (Completeness)

If $\models |\Gamma \rightarrow \Delta|$ *then* $\vdash_G \Gamma \Rightarrow \Delta$

Proof by contradiction. If not $\vdash_G \Gamma \Rightarrow \Delta$ then proof search fails.
Then there is an $\mathcal{A}$ such that $\mathcal{A}(\bigwedge \Gamma \rightarrow \bigvee \Delta) = 0$.
Therefore not $\models |\Gamma \rightarrow \Delta|$.

Natural Deduction

Recall: Natural Deduction rules

$$\frac{F \quad G}{F \wedge G}\ \wedge I \qquad \frac{F \wedge G}{F}\ \wedge E_1 \quad \frac{F \wedge G}{G}\ \wedge E_2$$

$$\frac{\begin{array}{c}[F]\\ \vdots \\ G\end{array}}{F \rightarrow G}\ \rightarrow I \qquad \frac{F \rightarrow G \quad F}{G}\ \rightarrow E$$

$$\frac{F}{F \vee G}\ \vee I_1 \quad \frac{G}{F \vee G}\ \vee I_2 \qquad \frac{F \vee G \quad \begin{array}{c}[F]\\ \vdots \\ H\end{array} \quad \begin{array}{c}[G]\\ \vdots \\ H\end{array}}{H}\ \vee E$$

$$\frac{\begin{array}{c}[F]\\ \vdots \\ \bot\end{array}}{\neg F}\ \neg I \qquad \frac{\neg F \quad F}{\bot}\ \neg E \qquad \frac{\begin{array}{c}[\neg F]\\ \vdots \\ \bot\end{array}}{F}\ \bot$$

Rules for quantifiers

$$\frac{F[y/x]}{\forall x\, F}\ \forall I(*) \qquad \frac{\forall x\, F}{F[t/x]}\ \forall E$$

$$\frac{F[t/x]}{\exists x\, F}\ \exists I \qquad \frac{\exists x\, F \qquad \begin{array}{c}[F[y/x]]\\ \vdots \\ H\end{array}}{H}\ \exists E(**)$$

$(*)$: $(y = x$ or $y \notin fv(F))$ and
y not free in an open assumption in the proof of $F[y/x]$

$(**)$: $(y = x$ or $y \notin fv(F))$ and
y not free in H or in an open assumption in the proof of the second premise, except for $F[y/x]$

Example of a proof

$$\forall x\,(\exists y\,P(y) \to Q(x)) \;\vdash_N\; \forall x\,\exists y\,(P(y) \to Q(x)):$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[P(z)]^3}{\exists y\,P(y)}\;\exists I{:}5 \qquad \dfrac{\forall x\,(\exists y\,P(y) \to Q(x))}{\exists y\,P(y) \to Q(z)}\;\forall E{:}6}{Q(z)}\;{\to}E{:}4}{P(z) \to Q(z)}\;{\to}I{:}3}{\exists y\,(P(y) \to Q(z))}\;\exists I{:}2}{\forall x\,\exists y\,(P(y) \to Q(x))}\;\forall I{:}1$$

Soundness

Theorem (Soundness)

If $\Gamma \vdash_N F$ *then* $\Gamma \models F$

Proof by induction of the depth of the proof tree for $\Gamma \vdash_N F$, with additional cases. We only consider one:

Case: rule applied to the root is

$$\dfrac{\exists x\, F \qquad \begin{matrix}[F[y/x]] \\ \vdots \\ H\end{matrix}}{H}\ \exists E(**)$$

$(**)$ y not free in H or in an open assumption in the proof of the second premise, except for $F[y/x]$.

To show: $\Gamma \models H$, i.e., for every $\mathcal{A}$, if $\mathcal{A} \models \Gamma$ then $\mathcal{A} \models H$.

IH: $\Gamma \models \exists x F$ and $F[y/x], \Gamma \models H$.

Soundness

To show: $\Gamma \models H$, i.e., for every $\mathcal{A}$, if $\mathcal{A} \models \Gamma$ then $\mathcal{A} \models H$.

IH: $\Gamma \models \exists x F$ and $F[y/x], \Gamma \models H$.

Pick $\mathcal{A}$ arbitrary.

$$
\begin{array}{lr}
\quad \mathcal{A} \models \Gamma & \\
\Rightarrow \mathcal{A} \models \exists x\, F & (\Gamma \models \exists x F \text{ by IH}) \\
\Rightarrow \mathcal{A}[u/x] \models F \text{ for some } u \in U^{\mathcal{A}} & (\text{semantics}) \\
\Rightarrow \mathcal{A}[u/y] \models F[y/x] & (y = x \text{ or } y \notin fv(F)) \\
\quad \text{and } \mathcal{A}[u/y] \models \Gamma & (y \text{ not free in } \Gamma) \\
\Rightarrow \mathcal{A}[u/y] \models H & (\text{transit. of } \models) \\
\Rightarrow \mathcal{A} \models H & (y \text{ not free in } H)
\end{array}
$$

Completeness

Theorem (ND can simulate SC)

If $\vdash_G \Gamma \Rightarrow \Delta$ *then* $\Gamma, \neg\Delta \vdash_N \bot$ *(where* $\neg\{F_1, \ldots\} = \{\neg F_1, \ldots\}$*).*

Proof by induction on (the depth of) $\vdash_G \Gamma \Rightarrow \Delta$. (Omitted.)

Corollary (Completeness of ND)

If $\Gamma \models F$ *then* $\Gamma \vdash_N F$*.*

Proof as before: apply the completeness of $\vdash_G$. (Omitted.)

Hilbert System

Recall: Hilbert System

Axioms:

$$F \to G \to F \tag{A1}$$

$$(F \to G \to H) \to (F \to G) \to F \to H \tag{A2}$$

$$F \to G \to F \wedge G \tag{A3}$$

$$F \wedge G \to F \tag{A4}$$

$$F \wedge G \to G \tag{A5}$$

$$F \to F \vee G \tag{A6}$$

$$G \to F \vee G \tag{A7}$$

$$F \vee G \to (F \to H) \to (G \to H) \to H \tag{A8}$$

$$(\neg F \to \bot) \to F \tag{A9}$$

Inference rule: modus ponens

$$\frac{F \to G \quad F}{G} \; {\to}E$$

New axioms and inference rule

Additional axioms:

$$\forall x\, F \rightarrow F[t/x]$$
$$F[t/x] \rightarrow \exists x\, F$$
$$\forall x (G \rightarrow F) \rightarrow (G \rightarrow \forall y\, F[y/x]) \quad (*)$$
$$\forall x (F \rightarrow G) \rightarrow (\exists y\, F[y/x] \rightarrow G) \quad (*)$$

$(*)$ where $x \notin fv(G)$ and $(y = x$ or $y \notin fv(F))$

Additional inference rule:

$$\frac{F}{\forall y\, F[y/x]} \; \forall I (*)$$

(*) provided x not free in the assumptions
and $(y = x$ or $y \notin fv(F))$.

Equivalence of Hilbert and ND

As before, with additional cases.