First-Order Logic

The Classical Decision Problem

Validity/satisfiability of arbitrary first-order formulas is undecidable.

What about subclasses of formulas?

Examples

$\forall x \exists y \, (P(x) \rightarrow P(y))$ Satisfiable? Resolution?

$\exists x \forall y \, (P(x) \rightarrow P(y))$ Satisfiable? Resolution?

The $\exists^*\forall^*$ class

Definition

The $\exists^*\forall^*$ class is the class of closed formulas of the form

$$\exists x_1 \dots \exists x_m \forall y_1 \dots \forall y_n \; F$$

where F is a quantifier-free formula that contains no function symbols of arity > 0.

This is also called the Bernays-Schönfinkel class.

Corollary

(Un)satisfiability is decidable for formulas in the $\exists^*\forall^*$ class.

Proof The Herbrand universe of $\exists^*\forall^*$-formulas is finite.

What if a formula is not in the $\exists^*\forall^*$ class?
Try to transform it into the $\exists^*\forall^*$ class!

Example

$$\forall y\,\exists x\,(P(x) \to Q(y)) \equiv \exists x\,\forall y\,(P(x) \to Q(y))$$

Heuristic transformation procedure (may or may not work):

1. Put formula into NNF.
2. Push all quantifiers into the formula as far as possible ("miniscoping").
3. Pull out $\exists$ first and $\forall$ afterwards.

Miniscoping

Perform the following transformations bottom-up, as long as possible:

- $(\exists x\ F) \ \equiv\ F$ if x does not occur free in F
- $\exists x\ (F \vee G) \ \equiv\ (\exists x\ F) \vee (\exists x\ G)$
- $\exists x\ (F \wedge G) \ \equiv\ (\exists x\ F) \wedge G$ if x is not free in G
- $\exists x\ F$ where F is a conjunction,
  x occurs free in every conjunct,
  and the DNF of F is of the form $F_1 \vee \cdots \vee F_n$, $n \geq 2$:
  $\exists x\, F \ \equiv\ \exists x\ (F_1 \vee \cdots \vee F_n)$.
- dual transformations for $\forall$ of all of the above.

Warning: Complexity!

Miniscoping

Example

$$
\begin{aligned}
& \exists x \left(P(x) \wedge \exists y \, (Q(y) \vee R(x))\right) \\
\equiv \quad & \exists x \left(P(x) \wedge (\exists y \, Q(y) \vee \exists y \, R(x))\right) \\
\equiv \quad & \exists x \left(P(x) \wedge (\exists y \, Q(y) \vee R(x))\right) \\
\equiv \quad & \exists x \left((P(x) \wedge \exists y \, Q(y)) \vee (P(x) \wedge R(x))\right) \\
\equiv \quad & \exists x \, (P(x) \wedge \exists y \, Q(y)) \vee \exists x \, (P(x) \wedge R(x)) \\
\equiv \quad & (\exists x \, P(x) \wedge \exists y \, Q(y)) \vee \exists x \, (P(x) \wedge R(x))
\end{aligned}
$$

The monadic class

Definition

A formula is monadic if it contains only unary (monadic) predicate symbols and no function symbol of arity > 0.

Examples

All men are mortal. Socrates is a man. Socrates is mortal.

The monadic class is decidable

Theorem

For every monadic formula, the heuristic transformation procedure yields an equisatisfiable $\exists^*\forall^*$-formula.

Proof Put into NNF and perform miniscoping.

The result has no nested quantifiers (Exercise!).

First pull out all $\exists$, then all $\forall$, and existentially quantify free variables.

The result is in the $\exists^*\forall^*$ class.

Corollary

(Un)satisfiability of monadic formulas is decidable.

The finite model property

Definition

A formula F has the finite model property (for satisfiability) if F has a model iff F has a finite model.

Theorem

If a class of formulas has the finite model property, satisfiability is decidable.

Proof. Two semi-decision procedures, one for unsatisfiability and one for satisfiability. The procedure for satisfiability searches systematically for a model through all structures with finite domain.

The finite model property

Another proof of decidability of satisfiability for monadic formulas:

Theorem

Monadic formulas have the finite model property.

Proof

We show: A satisfiable monadic formula F with k different monadic predicate symbols $P_1, \ldots, P_k$ has a model of size $\leq 2^k$.

Given a model $\mathcal{A}$ of F and $u, v \in U^{\mathcal{A}/\sim}$, define $u \sim v$ iff $P_i^{\mathcal{A}}(u) = P_i^{\mathcal{A}}(v)$ for every $1 \leq i \leq k$.

$\sim$ is a congruence (immediate consequence of the definition of congruence and the fact that all predicates are monadic).

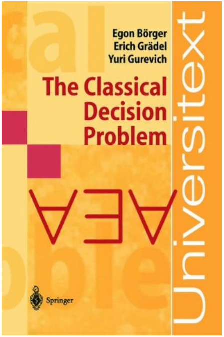
$\mathcal{A}_\sim$ (the quotient of $\mathcal{A}$ w.r.t. $\sim$) is also a model of F.

$|U_{\mathcal{A}/\sim}| \leq 2^k$, because an equivalence class $[u]_\sim$ is characterized by the bit-vector $(P_1^{\mathcal{A}}(u), \ldots, P_k^{\mathcal{A}}(u))$ of length k.

Classification by quantifier prefix of prenex form

There is a complete classification of decidable and undecidable classes of formulas based on

- the form of the quantifier prefix of the prenex form
- the arity of the predicate and function symbols allowed
- whether "=" is allowed or not.



A complete classification

Only formulas without function symbols of arity > 0,
no restrictions on predicate symbols.

Satisfiability is decidable:

$\exists^*\forall^*$ (Bernays, Schönfinkel 1928, Ramsey 1930)

$\exists^*\forall\exists^*$ (Ackermann 1928)

$\exists^*\forall^2\exists^*$ (Gödel 1932)

Satisfiability is undecidable:

$\forall^3\exists$ (Surányi 1959)

$\forall\exists\forall$ (Kahr, Moore, Wang 1962)

Why complete?

Famous mistake by Gödel: $\exists^*\forall^2\exists^*$ with "$=$" is undecidable
(Goldfarb 1984)