First-Order Logic
Compactness

[Harrison, Section 3.16]

More Herbrand Theory

Recall Gödel-Herbrand-Skolem:

Theorem

Let F be a closed formula in Skolem form. Then F is satisfiable iff its Herbrand expansion $E(F)$ is (propositionally) satisfiable.

$T(S)$: the set of all terms without variables constructed out of function symbols of S (plus a constant, if S contains none).
$E(S)$: set of all propositional formulas constructed by replacing the variables in the matrices of the formulas in S with terms from $T(S)$.
We have:

Theorem (1)

Let S be a set of closed formulas in Skolem form.
Then S is satisfiable iff $E(S)$ is (propositionally) satisfiable.

Proof: Show first that S is satisfiable iff it has a Herbrand model, and then that it is equivalent to the Herbrand expansion.

Transforming sets of formulas

Recall the transformation of single formulas into equisatisfiable Skolem form: close, RPF, skolemize

Theorem (2)

Let S be a countable set of closed formulas. Then we can transform it into an equisatisfiable set T of closed formulas in Skolem form. We call this transformation function skolem.

- Can all formulas in S be transformed in parallel?
- Why countable?

Transforming sets of formulas

Proof:

1. Put all formulas in S into RPF.

   Problem in Skolemization step: How do we generate new function symbols if all of them have been used already in S?

2. Rename all function symbols in S: $f_i^k \mapsto f_{2i}^k$

   The result: equisatisfiable countable set $\{F_0, F_1, \ldots\}$.

   Unused symbols: all f_{2i+1}^k

3. Skolemize the F_i one by one using the f_{2i+1}^k not used in the Skolemization of $F_0, \ldots, F_{i-1}$

Result is equisatisfiable with initial S.

Compactness

Theorem

Let S be a countable set of closed formulas.
If every finite subset of S is satisfiable, then S is satisfiable.

Proof every fin. $F \subseteq S$ is sat.
$\Rightarrow$ every fin. $F \subseteq \mathit{skolem}(S)$ is sat. by Theorem (2)
(fin. $F \subseteq \mathit{skolem}(S) \Rightarrow F \subseteq \mathit{skolem}(S_0)$ for some fin. $S_0 \subseteq S$)
$\Rightarrow$ for every fin. $F \subseteq \mathit{skolem}(S)$, $E(F)$ is prop. sat. by Theorem(1)
$\Rightarrow$ every fin. $F' \subseteq E(\mathit{skolem}(S))$ is prop. sat.
(there must exist a fin. $F \subseteq \mathit{skolem}(S)$ s.t. $F' \subseteq E(F)$)
$\Rightarrow$ $E(\mathit{skolem}(S))$ is prop. sat. by prop. compactness
$\Rightarrow$ $\mathit{skolem}(S)$ is sat. by Theorem (1)
$\Rightarrow$ S is sat. by Theorem (2)