First-Order Logic
Equality

Predicate logic with equality

Predicate logic

$+$

distinguished predicate symbol "=" of arity 2

Semantics: A structure $\mathcal{A}$ of predicate logic with equality always maps the predicate symbol = to the identity relation.

$$\mathcal{A}(=) \;=\; \{(d,d) \mid d \in U_{\mathcal{A}}\}$$

Expressivity

Fact

A structure is model of $\exists x \forall y\, x{=}y$ *iff its universe is a singleton.*

Theorem

Every satisfiable formula of predicate logic has a countably infinite model.

Proof Let F be satisfiable. Assume w.l.o.g. $F = \forall x_1 \dots \forall x_n\, F^*$ and the variables occurring in F^* are exactly $x_1, \dots, x_n$. (Bring F in closed Skolem form if needed.) Consider two cases:
$n = 0$. Exercise.
$n > 0$. Let $G = \forall x_1 \dots \forall x_n\, F^*[f(x_1)/x_1]$, where f does not occur in F^*. G is satisfiable (why?), and if G has a model M with universe U, then F has a model with universe $\{f^M(u) \mid u \in U\}$. By the fundamental theorem, G has a model with universe $T(G)$, which is countable infinite. So F also has a model with countably infinite universe.

Modeling equality

We assign to every formula F of predicate logic with equality a formula E_F of predicate logic.

Let Eq be a predicate symbol that does not occur in F.
E_F is the conjunction of the following formulas:

$$\forall x \, Eq(x, x)$$

$$\forall x \, \forall y \, (Eq(x, y) \rightarrow Eq(y, x))$$

$$\forall x \, \forall y \, \forall z \, ((Eq(x, y) \wedge Eq(y, z)) \rightarrow Eq(x, z))$$

For every function symbol f in F of arity n and every $1 \leq i \leq n$:

$$\forall x_1 \ldots \forall x_n \, \forall y \, (Eq(x_i, y) \rightarrow Eq(f(x_1, \ldots, x_i, \ldots x_n), f(x_1, \ldots, y, \ldots, x_n)))$$

For every predicate symbol P in F of arity n and every $1 \leq i \leq n$:

$$\forall x_1 \ldots \forall x_n \, \forall y (Eq(x_i, y) \rightarrow (P(x_1, \ldots, x_i, \ldots, x_n) \leftrightarrow P(x_1, \ldots, y, \ldots, x_n)))$$

E_F expresses that Eq is a *congruence (relation)* on the symbols of the formula F.

Quotient structure

Definition

Let $\mathcal{A}$ be a structure and let $\sim$ be an equivalence relation on $U_{\mathcal{A}}$ that is a congruence for all the predicate and function symbols defined by $I_{\mathcal{A}}$. The quotient structure $\mathcal{A}/_{\sim}$ is defined as follows:

- $U_{\mathcal{A}/_{\sim}} = \{[u]_{\sim} \mid u \in U_{\mathcal{A}}\}$ where $[u]_{\sim} = \{v \in U_{\mathcal{A}} \mid u \sim v\}$
- For every function symbol f defined by $I_{\mathcal{A}}$:
  $f^{\mathcal{A}/\sim}([d_1]_{\sim}, \ldots, [d_n]_{\sim}) = [f^{\mathcal{A}}(d_1, \ldots, d_n)]_{\sim}$
- For every predicate symbol P defined by $I_{\mathcal{A}}$:
  $P^{\mathcal{A}/\sim}([d_1]_{\sim}, \ldots, [d_n]_{\sim}) = P^{\mathcal{A}}(d_1, \ldots, d_n)$
- For every variable x defined by $I_{\mathcal{A}}$: $x^{\mathcal{A}/\sim} = [x^{\mathcal{A}}]_{\sim}$

Lemma

$\mathcal{A}/_{\sim}(t) = [\mathcal{A}(t)]_{\sim}$ *and* $\mathcal{A}/_{\sim}(F) = \mathcal{A}(F)$

Theorem

The formulas F and $E_F \wedge F[Eq/{=}]$ are equisatisfiable.

Proof

($\Leftarrow$): If $E_F \wedge F[Eq/{=}]$ is satisfiable, then F is satisfiable.

Assume $\mathcal{A} \models E_F \wedge F[Eq/{=}]$. Then $Eq^{\mathcal{A}}$ is a congruence.

Define $\mathcal{B} = \mathcal{A}/_{Eq^{\mathcal{A}}}$ (extended with $=$ interpreted as identity).

We prove $\mathcal{B}(F) = 1$.

Claim 1: $\mathcal{B} \models F[Eq/{=}]$. Follows from the Lemma.

Claim 2: $Eq^{\mathcal{B}}$ is the identity.

$$\begin{array}{lll} & Eq^{\mathcal{B}}([a]_{Eq^{\mathcal{A}}}, [a']_{Eq^{\mathcal{A}}}) & \\ = & Eq^{\mathcal{A}}(a, a') & \text{(Def. of quotient structure)} \\ = & ([a]_{Eq^{\mathcal{A}}} = [a']_{Eq^{\mathcal{A}}}) & \text{(Def. of equivalence class)} \end{array}$$

We have: $\mathcal{B}(F) \overset{(2)}{=} \mathcal{B}(F[Eq/{=}]) \overset{(1)}{=} 1$.

($\Leftarrow$): If F is satisfiable., then $E_F \wedge F[Eq/{=}]$ is satisfiable.

Any model of F yields a model of $E_F \wedge F[Eq/{=}]$ by interpreting Eq as equality.