First-Order Logic
Resolution

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We upgrade resolution to make it work for predicate logic.

Recall: resolution in propositional logic

Resolution step:

$$\begin{array}{ccc} \{L_1, \ldots, L_n, A\} & & \{L'_1, \ldots, L'_m, \neg A\} \\ & \{L_1, \ldots, L_n, L'_1, \ldots, L'_m\} & \end{array}$$

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\end{tikzcd}
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Resolution graph:

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\{\neg A, B\} \arrow[dr, dash] & & \{A\} \arrow[dl, dash] & & \{\neg B\} \arrow[ddl, dash] \\
& \{B\} \arrow[drr, dash] & & & \\
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A set of clauses is unsatisfiable iff the empty clause can be derived.

Adapting Gilmore's Algorithm

Gilmore's Algorithm:

Let F be a closed formula in Skolem form
and let $F_1, F_2, F_3, \ldots$ be an enumeration of $E(F)$.

$n := 0$;
repeat $n := n + 1$
until $(F_1 \wedge F_2 \wedge \ldots \wedge F_n)$ is unsatisfiable;
  – *this can be checked with any calculus for propositional logic*
return "unsatisfiable"

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"any calculus" $\rightsquigarrow$ use resolution for the unsatisfiability test

Terminology

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Literal/clause/CNF is defined as for propositional logic
but with the atomic formulas of predicate logic.

A ground term/formula/etc is a term/formula/etc
that does not contain any variables.

An instance of a term/formula/etc
is the result of applying a substitution to a term/formula/etc.

A ground instance
is an instance that does not contain any variables.

Clause Herbrand expansion

Let $F = \forall y_1 \dots \forall y_n \, F^*$ be a closed formula in Skolem form with F^* in CNF, and let $C_1, \dots, C_m$ be the clauses of F^*.

The clause Herbrand expansion of F is the set of ground clauses

$$CE(F) = \bigcup_{i=1}^{m} \{ C_i[t_1/y_1] \dots [t_n/y_n] \mid t_1, \dots, t_n \in T(F) \}$$

Lemma

$CE(F)$ is unsatisfiable iff $E(F)$ is unsatisfiable.

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$CE(F)$ is unsatisfiable iff $E(F)$ is unsatisfiable.

Proof. Informally speaking, "$CE(F) \equiv E(F)$".

Ground resolution algorithm

Let F be a closed formula in Skolem form with F^* in CNF.

Let $C_1, C_2, C_3, \ldots$ be an enumeration of $CE(F)$.

$n := 0$;
$S := \emptyset$;
repeat
  $n := n + 1$;
  $S := S \cup \{C_n\}$;
until $S \vdash_{Res} \Box$

return "unsatisfiable"

Ground resolution algorithm

Note: For example, $CE(F)$ can be enumerated according to the size of the substitutions.

Let $F = \forall y_1 \ldots \forall y_n\, F^*$ and let $C_1, \ldots, C_m$ be the clauses of F^*. For every $s \geq 0$, define

$$\mathcal{C}_s = \bigcup_{i=1}^{m} \left\{ C_i[t_1/y_1]\ldots[t_n/y_n] \;\middle|\; \begin{array}{c} t_1, \ldots, t_n \in T(F) \\ \text{and} \\ |t_1| + \cdots + |t_n| = s \end{array} \right\}$$

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$\mathcal{C}_s$ is finite for every $s \geq 0$ and $CE(F) = \bigcup_{s=0}^{\infty} \mathcal{C}_s$.

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Note: The search for $\square$ can be performed incrementally every time S is extended, keeping the clauses generated in previous steps.

Ground resolution theorem

The correctness of the ground resolution algorithm can be rephrased as follows:

Theorem

A formula $F = \forall y_1 \ldots \forall y_n \; F^*$ with F^* in CNF is unsatisfiable iff there is a sequence of ground clauses $C_1, \ldots, C_m = \Box$ such that for every $i = 1, \ldots, m$

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- *either C_i is a ground instance of a clause $C \in F^*$, i.e. $C_i = C[t_1/y_1] \ldots [t_n/y_n]$ where $t_1, \ldots, t_n \in T(F)$,*
- *or C_i is a resolvent of two clauses C_a, C_b with $a < i$ and $b < i$*

Beyond ground resolution: Intuition

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$$F^* = \{\, \{P(x)\}\,,\, \{\neg P(f(g(b,y))), Q(y)\}\,,\, \{\neg Q(g(f(z),f(z)))\}\,\}.$$

The algorithm can derive $\Box$ from just three ground clauses:

$$\{P(f(g(b,g(f(a),f(a)))))\}$$
$$\{\neg P(f(g(b,g(f(a),f(a))))), Q(g(f(a),f(a)))\}$$
$$\{\neg Q(g(f(a),f(a)))\}$$

Blind enumeration will generate the third clause early on, but it will only generate the first two after many (many!) superfluous clauses.

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When resolving the first two clauses, delay the choice of substitution for x.

Commit only to replacing x by

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For this:

- Allow substitutions with variables: $[f(g(b,y))/x]$.
- Apply substitutions only to two clauses that enable a new resolution step.

Beyond ground resolution: Intuition

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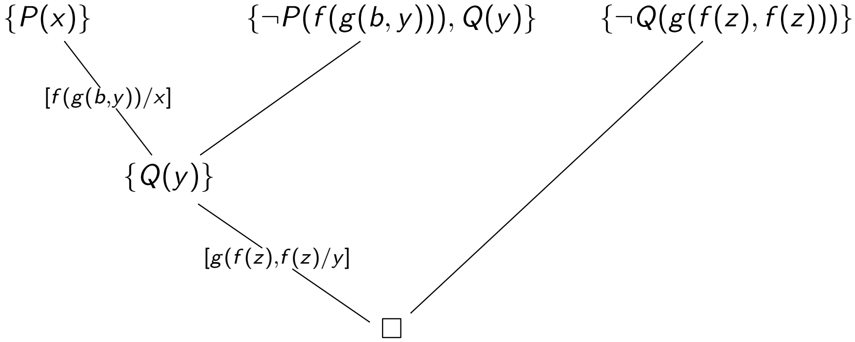

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Substitutions are functions from variables to terms:
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Substitutions are functions. Therefore

$$\sigma_1 = \sigma_2 \quad \text{iff} \quad x\sigma_1 = x\sigma_2 \text{ for all variables } x$$

Substitutions as functions

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Substitutions are defined to have finite domain, and so every substitution can be written as a

simultaneous substitution $[t_1/x_1, \ldots, t_n/x_n]$.

Unifier and most general unifier

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Let $\mathbf{L} = \{L_1, \ldots, L_k\}$ be a set of literals.

A substitution σ is a unifier of $\mathbf{L}$ if

$$L_1\sigma = L_2\sigma = \cdots = L_k\sigma$$

i.e. if $|\mathbf{L}\sigma| = 1$, where $\mathbf{L}\sigma = \{L_1\sigma, \ldots, L_k\sigma\}$.

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Exercise

Unifiable?			Yes	No
	$P(f(x))$	$P(g(y))$		
	$P(x)$	$P(f(y))$		
	$P(x)$	$P(f(x))$		
	$P(x, f(y))$	$P(f(u), f(z))$		
	$P(x, f(x))$	$P(f(y), y)$		
	$P(x, g(x), g^2(x))$	$P(f(z), w, g(w))$		
$P(x, f(y))$	$P(g(y), f(a))$	$P(g(a), z)$		

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if none of the two characters at that position is a variable

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  else let x be the variable and t the term starting at that position

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Unification algorithm

Example

$\neg$`P(f(z,g(a,y)), h(z))`,

$\neg$`P(f(f(u,v),w), h(f(a,b)))`

Correctness of the unification algorithm

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Lemma

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Proof If $\mathbf{L}$ is non-unifiable then the algorithm can never exit the loop normally.

Correctness/completeness of the unification algorithm

Lemma

If $\mathbf{L}$ *is unifiable then the algorithm returns the mgu of* $\mathbf{L}$
(and so in particular every unifiable set $\mathbf{L}$ *has an mgu).*

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Lemma

If **L** *is unifiable then the algorithm returns the mgu of* **L** *(and so in particular every unifiable set* **L** *has an mgu).*

Proof Assume **L** is unifiable and let n be the number of iterations of the loop on input **L**.

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We prove for every $0 \leq i \leq n$:

(a) If $1 \leq i$, the i-th iteration does not return "non-unifiable".

(b) For every unifier σ' of $\mathbf{L}$ there is a substitution δ_i such that $\sigma' = \sigma_i \, \delta_i$.

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(b) For every unifier σ' of $\mathbf{L}$ there is a substitution δ_i such that $\sigma' = \sigma_i \, \delta_i$.

By (a) the algorithm exits the loop normally after n iterations.
By (b) it returns a most general unifier.

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For (b): Let σ' be a unifier of $\mathbf{L}$.

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Example

$C_1 = \{\ P(x),\ Q(x),\ P(g(y))\ \}$ and $C_2 = \{\ \neg P(x),\ R(f(x), a)\ \}$

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How many resolvents are there?

C_1	C_2	Resolvents
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Why renaming?

Example

$\forall x(P(x) \land \neg P(f(x)))$

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Completeness Does unsatisfiability of F imply $F \vdash_{Res} \Box$?

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Derive $\square$ from the following clauses:

1. $\{\neg P(x), Q(x), R(x, f(x))\}$
2. $\{\neg P(x), Q(x), S(f(x))\}$
3. $\{T(a)\}$
4. $\{P(a)\}$
5. $\{\neg R(a, z), T(z)\}$
6. $\{\neg T(x), \neg Q(x)\}$
7. $\{\neg T(y), \neg S(y)\}$

Correctness of Resolution for First-Order Logic

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The universal closure of a formula H with free variables $x_1, \ldots, x_n$:

$$\forall H \;=\; \forall x_1 \forall x_2 \ldots \forall x_n H$$

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$$\forall C_a, \forall C_b \models \forall C_m \qquad (*)$$

Thus $\forall F^* \models \forall C_m$ because $\forall F^* \models \forall C_a$ and $\forall F^* \models \forall C_b$ by IH.

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Proof of $(*)$: Assume $\mathcal{A}(\forall C_a) = \mathcal{A}(\forall C_b) = 1$. $\qquad (**)$
We prove $\mathcal{A}(\forall C_m) = 1$ by contradiction. Assume $\mathcal{A}(\forall C_m) = 0$.

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Thus $\forall F^* \models \forall C_m$ because $\forall F^* \models \forall C_a$ and $\forall F^* \models \forall C_b$ by IH.

Proof of $(*)$: Assume $\mathcal{A}(\forall C_a) = \mathcal{A}(\forall C_b) = 1$. $\qquad (**)$
We prove $\mathcal{A}(\forall C_m) = 1$ by contradiction. Assume $\mathcal{A}(\forall C_m) = 0$.
By def.

$$\begin{aligned} C_m &= ((C_a - \{L_1, \ldots\}) \cup (C_b\rho - \{L'_1, \ldots\}))\sigma \\ &= (C_a\sigma - \{L\}) \cup (C_b\rho\sigma - \{\overline{L}\}) \end{aligned}$$

$\Rightarrow \mathcal{A}'(C_m) = 0$ where $\mathcal{A}' = \mathcal{A}[u_1/x_1, \ldots]$ for some $u_i \in U_{\mathcal{A}}$
$\Rightarrow \mathcal{A}'(C_a\sigma - \{L\}) = \mathcal{A}'(C_b\rho\sigma - \{\overline{L}\}) = 0$
$\Rightarrow \mathcal{A}'(L) = \mathcal{A}'(\overline{L}) = 1$ becs. $\mathcal{A}'(C_a\sigma) = \mathcal{A}'(C_b\rho\sigma) = 1$ becs. $(**)$
Contradiction

Completeness: The idea

Simulate ground resolution because that is complete

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Lift the resolution proof from the ground resolution proof

Lifting Lemma

Lifting Lemma

C_1 C_2

Let C_1, C_2 be two clauses

Lifting Lemma

Let C_1, C_2 be two clauses and
let C_1', C_2' be two ground instances

$$\begin{array}{ccc} C_1 & \qquad & C_2 \\ \downarrow & & \downarrow \\ C_1' & & C_2' \end{array}$$

$\rightarrow$: Substitution

Lifting Lemma

Let C_1, C_2 be two clauses and let C_1', C_2' be two ground instances with (propositional) resolvent R'.

$$
\begin{tikzcd}
C_1 \arrow[d] & & C_2 \arrow[d] \\
C_1' \arrow[dr, dash] & & C_2' \arrow[dl, dash] \\
& R' &
\end{tikzcd}
$$

$\rightarrow$: Substitution

—: Resolution

Lifting Lemma

Let C_1, C_2 be two clauses and
let C_1', C_2' be two ground instances
with (propositional) resolvent R'.

Then there is a resolvent R of C_1, C_2

$$\begin{tikzcd}
C_1 \arrow[d] \arrow[dr, dash, dashed] & & C_2 \arrow[d] \arrow[dl, dash, dashed] \\
C_1' \arrow[dr, dash] & R & C_2' \arrow[dl, dash] \\
& R' &
\end{tikzcd}$$

$\rightarrow$: Substitution
—: Resolution

Lifting Lemma

Let C_1, C_2 be two clauses and
let C_1', C_2' be two ground instances
with (propositional) resolvent R'.

Then there is a resolvent R of C_1, C_2
such that R' is a ground instance of R.

$$
\begin{tikzcd}
C_1 \arrow[d] \arrow[dr, dash, dashed] & & C_2 \arrow[d] \arrow[dl, dash, dashed] \\
C_1' \arrow[dr, dash] & R \arrow[d, dashed] & C_2' \arrow[dl, dash] \\
& R' &
\end{tikzcd}
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$\rightarrow$: Substitution
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Lifting Lemma: example

$\{\neg P(f(x)), Q(x)\}$ $\{P(f(g(y)))\}$

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$$\{\neg P(f(x)), Q(x)\} \qquad\qquad \{P(f(g(y)))\}$$

$$\Big\downarrow [g(a)/x] \qquad\qquad\qquad \Big\downarrow [a/y]$$

Lifting Lemma: example

$$\begin{array}{c} \{\neg P(f(x)), Q(x)\} \\ \Big\downarrow [g(a)/x] \\ \{\neg P(f(g(a))), Q(g(a))\} \end{array} \qquad \begin{array}{c} \{P(f(g(y)))\} \\ \Big\downarrow [a/y] \\ \{P(f(g(a)))\} \end{array}$$

Lifting Lemma: example

$$
\begin{array}{ccccc}
\{\neg P(f(x)), Q(x)\} & & & & \{P(f(g(y)))\} \\
\downarrow [g(a)/x] & & & & \downarrow [a/y] \\
\{\neg P(f(g(a))), Q(g(a))\} & & & & \{P(f(g(a)))\} \\
& \searrow & & \swarrow & \\
& & \{Q(g(a))\} & &
\end{array}
$$

Lifting Lemma: example

$$
\begin{tikzcd}
\{\neg P(f(x)), Q(x)\} \arrow[d, "{[g(a)/x]}"] \arrow[dr, dash] & & \{P(f(g(y)))\} \arrow[d, "{[a/y]}"] \arrow[dl, dash] \\
\{\neg P(f(g(a))), Q(g(a))\} \arrow[dr, dash] & \{Q(g(y))\} & \{P(f(g(a)))\} \arrow[dl, dash] \\
& \{Q(g(a))\} &
\end{tikzcd}
$$

Lifting Lemma: example

$$\begin{tikzcd}
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\end{tikzcd}$$

Proof of Lifting Lemma.

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(1) C_1', C_2' are ground instances of C_1, C_2

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(3) Let ρ be a renaming s.t. C_1 and $C_2\rho$ have no common variables

Proof of Lifting Lemma.

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(3) Let ρ be a renaming s.t. C_1 and $C_2\rho$ have no common variables

(1) $\Rightarrow$ C_2' is a ground instance of $C_2\rho$. Thus there are σ_1, σ_2 s.t.

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$C_1' = C_1\sigma_1$

Proof of Lifting Lemma.

(1) C_1', C_2' are ground instances of C_1, C_2

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We prove that R' is an instance of a resolvent of C_1 and C_2

(3) Let ρ be a renaming s.t. C_1 and $C_2\rho$ have no common variables

(1) $\Rightarrow$ C_2' is a ground instance of $C_2\rho$. Thus there are σ_1, σ_2 s.t.

$C_1' = C_1\sigma_1$ and $C_2' = C_2\rho\sigma_2$

Proof of Lifting Lemma.

(1) C'_1, C'_2 are ground instances of C_1, C_2
(2) R' is propositional resolvent of C'_1 and C'_2
We prove that R' is an instance of a resolvent of C_1 and C_2

(3) Let ρ be a renaming s.t. C_1 and $C_2\rho$ have no common variables
(1) $\Rightarrow$ C'_2 is a ground instance of $C_2\rho$. Thus there are σ_1, σ_2 s.t.
$C'_1 = C_1\sigma_1$ and $C'_2 = C_2\rho\sigma_2$ and $dom(\sigma_1) \cap dom(\sigma_2) = \emptyset$

Proof of Lifting Lemma.
(1) C_1', C_2' are ground instances of C_1, C_2
(2) R' is propositional resolvent of C_1' and C_2'
We prove that R' is an instance of a resolvent of C_1 and C_2

(3) Let ρ be a renaming s.t. C_1 and $C_2\rho$ have no common variables
(1) $\Rightarrow$ C_2' is a ground instance of $C_2\rho$. Thus there are σ_1, σ_2 s.t.
$C_1' = C_1\sigma_1$ and $C_2' = C_2\rho\sigma_2$ and $dom(\sigma_1) \cap dom(\sigma_2) = \emptyset$
$\Rightarrow$ $C_1' = C_1\sigma$

Proof of Lifting Lemma.

(1) C_1', C_2' are ground instances of C_1, C_2

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(2) $\Rightarrow$ $R' = (C_1' - \{L\}) \cup (C_2' - \{\overline{L}\})$

Proof of Lifting Lemma.

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s.t. σ is a unifier of $\{\overline{L_1}, \dots, L_1', \dots\}$

Proof of Lifting Lemma.

(1) C_1', C_2' are ground instances of C_1, C_2

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Let σ_0 be an mgu of M

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s.t. σ is a unifier of $\{\overline{L_1}, \dots, L_1', \dots\} =: M$.

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Proof of Lifting Lemma.

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$\Rightarrow$ A resolvent of C_1 and C_2:

Proof of Lifting Lemma.

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s.t. σ is a unifier of $\{\overline{L_1}, \dots, L_1', \dots\} =: M$.
Let σ_0 be an mgu of M and let $\sigma = \sigma_0\delta$ for some δ
$\Rightarrow$ A resolvent of C_1 and C_2:
$R := ((C_1 - \{L_1, \dots\}) \cup (C_2\rho - \{L_1', \dots\}))\sigma_0$

Proof of Lifting Lemma.

(1) C_1', C_2' are ground instances of C_1, C_2

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$R\delta$

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$\Rightarrow$ A resolvent of C_1 and C_2:

$R := ((C_1 - \{L_1, \dots\}) \cup (C_2\rho - \{L_1', \dots\}))\sigma_0$

$R\delta = ((C_1 - \{L_1, \dots\}) \cup (C_2\rho - \{L_1', \dots\}))\sigma$

Proof of Lifting Lemma.

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s.t. σ is a unifier of $\{\overline{L_1}, \dots, L_1', \dots\} =: M$.

Let σ_0 be an mgu of M and let $\sigma = \sigma_0\delta$ for some δ

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(1) C_1', C_2' are ground instances of C_1, C_2

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We prove that R' is an instance of a resolvent of C_1 and C_2

(3) Let ρ be a renaming s.t. C_1 and $C_2\rho$ have no common variables

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If F is unsatisfiable then $F^* \vdash_{Res} \Box$.

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Resolution Theorem for First-Order Logic

Theorem

Let F be a closed formula in Skolem form with matrix F^* in CNF.
Then F is unsatisfiable iff $F^* \vdash_{Res} \square$.

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  ($\Rightarrow F$ is satisfiable)

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But: Completeness must be preserved!