First-Order Logic

Normal Forms

Abbreviations

We return to the abbreviations used in connection with resolution:

$F_1 \rightarrow F_2$ abbreviates $\neg F_1 \lor F_2$

$\top$ abbreviates $P_1^0 \lor \neg P_1^0$

$\bot$ abbreviates $P_1^0 \land \neg P_1^0$

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- Similarly for subsitutions in terms:
  $u[t/x]$ is the result of replacing x by t in term u.

  Example: $(f(x))[g(x)/x] \ = \ f(g(x))$

Variable capture

If a term t of F contains a bound occurrence of a variable, substitution may lead to variable capture:

$$(\forall x \; P(x, y))[f(x)/y] \; = \; \forall x \; P(x, f(x))$$

Variable capture must be avoided

Substitution lemmas

Lemma

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Lemma (Substitution Lemma)

If t contains no variable bound in F then

$$\mathcal{A}(F[t/x]) \;=\; (\mathcal{A}[\mathcal{A}(t)/x])(F).$$

Proof by structural induction on F with the help of the lemma on terms.

Warning

The notation $.[./.]$ is heavily overloaded:

Substitution in syntactic objects

$F[G/A]$ in propositional logic
$F[t/x]$
$u[t/x]$ where u is a term

Function update

$\mathcal{A}[v/A]$ where $\mathcal{A}$ is a propositional assignment
$\mathcal{A}[d/x]$ where $\mathcal{A}$ is a structure and $d \in U_{\mathcal{A}}$

Overall goal

Transform any formula $F$of length m into a closed formula

$$\forall x_1 \dots \forall x_n \, G \quad \text{where } G \text{ is quantifier-free,}$$

of lengt $O(m)$ that is equisatisfiable with F.

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Let $F = Qx\, G$ *be a formula where* $Q \in \{\forall, \exists\}$.
Let y *be a variable that does not occur in* G.
Then $F \equiv Qy\, G[y/x]$.

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Then $F \equiv Qy\, G[y/x]$.

Lemma

Every formula is equivalent to a rectified formula.

Example

$$\forall x\, P(x, y) \wedge \exists x \exists y\, Q(x, y) \;\equiv\; \forall x'\, P(x', y) \wedge \exists x \exists y'\, Q(x, y')$$

Prenex form

Definition

A formula is in prenex form if it has the form

$$Q_1 y_1 \dots Q_n y_n \, F$$

where $Q_i \in \{\exists, \forall\}$, $n \geq 0$, and F is quantifier-free.

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Every formula is equivalent to a rectified formula in prenex form (a formula in **RPF***).*

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Proof First construct an equivalent rectified formula.
Then pull the quantifiers to the front using the following equivalences from left to right as long as possible:

$$
\begin{aligned}
\neg\forall x\, F &\equiv \exists x\, \neg F \\
\neg\exists x\, F &\equiv \forall x\, \neg F \\
Qx\, F \wedge G &\equiv Qx\, (F \wedge G) \\
F \wedge Qx\, G &\equiv Qx\, (F \wedge G) \\
Qx\, F \vee G &\equiv Qx\, (F \vee G) \\
F \vee Qx\, G &\equiv Qx\, (F \vee G)
\end{aligned}
$$

For the last four rules note that the formula is rectified!

Skolem form

The Skolem form of a formula F in RPF is the result of applying the following algorithm to F:

while F contains an existential quantifier **do**

Let $F = \forall y_1 \forall y_2 \ldots \forall y_n \exists z\ G$
(the block of universal quantifiers may be empty)

Let f be a fresh function symbol of arity n
that does not occur in F.

$$F := \forall y_1 \forall y_2 \ldots \forall y_n\ G[f(y_1, y_2, \ldots, y_n)/z]$$

i.e. remove the outermost existential quantifier in F and replace every occurrence of z in G by $f(y_1, y_2, \ldots, y_n)$

Example

$\exists x \, \forall y \, \exists z \, \forall u \, \exists v \, P(x, y, z, u, v) \equiv$

Exercise

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	R	P	S
$\forall x\,(T(x) \vee C(x) \vee D(x))$			

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$\neg\exists x\,C(x) \leftrightarrow \forall x\,\neg C(x)$			

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$\Rightarrow$ for all $u \in U_{\mathcal{A}}$, $\mathcal{A}(\exists z\, G) = 1$

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Proof Show: Every iteration produces an equisatisfiable formula.
Let (for simplicity) $F = \forall y\, \exists z\, G$ and $F' = \forall y\, G[f(y)/z]$.

2. If F has a model, so does F'

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Let $\mathcal{A}'$ be $\mathcal{A}$ extended with a definition of f: $f^{\mathcal{A}'}(u) := v$, where v is chosen as in $(*)$.

$\Rightarrow \mathcal{A}'(F') = 1$ because for all $u \in U_{\mathcal{A}}$:

$$\begin{aligned}
\mathcal{A}'[u/y](G[f(y)/z]) &= \mathcal{A}'[u/y][f^{\mathcal{A}'[u/y]}(u)/z](G) && \text{(subs. lemma)}\\
&= \mathcal{A}'[u/y][f^{\mathcal{A}'}(u)/z](G) && \text{(def. of } \mathcal{A}'\text{)}\\
&= \mathcal{A}'[u/y][v/z](G) = 1 && \text{(def. of } \mathcal{A}' \text{ and (*))}
\end{aligned}$$

Summary: conversion to Skolem form

Input: a formula F

Output: a rectified, closed formula in Skolem form $\forall y_1 \ldots \forall y_k \ G$, where G is quantifier-free, that is equisatisfiable with F.

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   The result is a formula F_1 equivalent to F.

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2. Let $y_1, y_2, \ldots, y_n$ be the variables occurring free in F_1.
   Produce the formula $F_2 = \exists y_1 \exists y_2 \ldots \exists y_n \, F_1$.
   F_2 is equisatisfiable with F_1, rectified and closed.

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3. Produce a formula F_3 in RPF equivalent to F_2.

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   F_2 is equisatisfiable with F_1, rectified and closed.
3. Produce a formula F_3 in RPF equivalent to F_2.
4. Eliminate the existential quantifiers in F_3
   by transforming F_3 into its Skolem form F_4.
   The formula F_4 is equisatisfiable with F_3.

Exercise

Convert into Skolem form $F = \forall x\, P(y, f(x, y)) \vee \neg\forall y\, Q(g(x), y)$