First-Order Predicate Logic
Basics

Syntax of predicate logic: terms

A variable is a symbol of the form x_i where $i = 1, 2, 3 \ldots$.

A function symbol is of the form f_i^k where $i = 1, 2, 3 \ldots$ and $k = 0, 1, 2 \ldots$.

A predicate symbol is of the form P_i^k where $i = 1, 2, 3 \ldots$ and $k = 0, 1, 2 \ldots$.

We call i the index and k the arity of the symbol. Function

symbols of arity 0 are called constant symbols.
Instead of $f_i^0()$ we write f_i^0.

Syntax of predicate logic: terms

A variable is a symbol of the form x_i where $i = 1, 2, 3 \ldots$.

A function symbol is of the form f_i^k where $i = 1, 2, 3 \ldots$ and $k = 0, 1, 2 \ldots$.

A predicate symbol is of the form P_i^k where $i = 1, 2, 3 \ldots$ and $k = 0, 1, 2 \ldots$.

We call i the index and k the arity of the symbol. Function

symbols of arity 0 are called constant symbols.
Instead of $f_i^0()$ we write f_i^0.

Terms are inductively defined as follows:

1. Variables are terms.
2. If f is a function symbol of arity k and $t_1, \ldots, t_k$ are terms then $f(t_1, \ldots, t_k)$ is a term.

Syntax of predicate logic: formulas

If P is a predicate symbol of arity k and $t_1, \ldots, t_k$ are terms then $P(t_1, \ldots, t_k)$ is an atomic formula.
If $k = 0$ we write P instead of $P()$.

Formulas (of predicate logic) are inductively defined as follows:

- Every atomic formula is a formula.
- If F is a formula, then $\neg F$ is also a formula.
- If F and G are formulas, then $F \wedge G$, $F \vee G$ and $F \rightarrow G$ are also formulas.
- If x is a variable and F is a formula, then $\forall x\ F$ and $\exists x\ F$ are also formulas. The symbols $\forall$ and $\exists$ are called the universal and the existential quantifier.

Syntax trees and subformulas

Syntax trees are defined as before, extended with the following trees for $\forall x F$ and $\exists x F$:

$$\begin{array}{c} \forall x \\ | \\ F \end{array} \qquad\qquad \begin{array}{c} \exists x \\ | \\ F \end{array}$$

Subformulas again correspond to subtrees.

Sructural induction of formulas

Like for propositional logic but

- Different base case: $\mathcal{P}(P(t_1, \ldots, t_k))$
- Two new induction steps:

  prove $\mathcal{P}(\forall x\ F)$ under the induction hypothesis $\mathcal{P}(F)$
  prove $\mathcal{P}(\exists x\ F)$ under the induction hypothesis $\mathcal{P}(F)$

Naming conventions

$x, y, z, \ldots$	instead of	$x_1, x_2, x_3, \ldots$
$a, b, c, \ldots$	for	constant symbols
$f, g, h, \ldots$	for	function symbols of arity > 0
$P, Q, R, \ldots$	instead of	P_i^k

Precedence of quantifiers

Quantifiers have the same precedence as $\neg$

Example

$\forall x\ P(x) \wedge Q(x)$ abbreviates $(\forall x\ P(x)) \wedge Q(x)$

not $\forall x\ (P(x) \wedge Q(x))$

Similarly for $\vee$ etc.

[This convention is not universal]

Free and bound variables, closed formulas

A variable x occurs in a formula F if it occurs in some atomic subformula of F.

Free and bound variables, closed formulas

A variable x occurs in a formula F if it occurs in some atomic subformula of F.

An occurrence of a variable in a formula is either free or bound.

An occurrence of x in F is bound if it occurs in some subformula of F of the form $\exists x G$ or $\forall x G$; the smallest such subformula is the scope of the occurrence.
Otherwise the occurrence is free.

Free and bound variables, closed formulas

A variable x occurs in a formula F if it occurs in some atomic subformula of F.

An occurrence of a variable in a formula is either free or bound.

An occurrence of x in F is bound if it occurs in some subformula of F of the form $\exists x G$ or $\forall x G$; the smallest such subformula is the scope of the occurrence.
Otherwise the occurrence is free.

A formula without any free occurrence of any variable is closed.

Example

$\forall x\ P(x) \rightarrow \exists y\ Q(a, x, y)$

Exercise

	Closed?
$\forall x\ P(a)$	

	Formula?

Exercise

	Closed?
$\forall x \; P(a)$	Y

	Formula?

Exercise

	Closed?
$\forall x\ P(a)$	Y
$\forall x \exists y\ (Q(x,y) \vee R(x,y))$	

	Formula?

Exercise

	Closed?
$\forall x \; P(a)$	Y
$\forall x \exists y \; (Q(x,y) \vee R(x,y))$	Y

	Formula?

Exercise

	Closed?
$\forall x\ P(a)$	Y
$\forall x \exists y\ (Q(x,y) \vee R(x,y))$	Y
$\forall x\ Q(x,x) \to \exists x\ Q(x,y)$	

	Formula?

Exercise

	Closed?
$\forall x \; P(a)$	Y
$\forall x \exists y \; (Q(x,y) \vee R(x,y))$	Y
$\forall x \; Q(x,x) \to \exists x \; Q(x,y)$	N

	Formula?

Exercise

	Closed?
$\forall x\ P(a)$	Y
$\forall x \exists y\ (Q(x,y) \vee R(x,y))$	Y
$\forall x\ Q(x,x) \to \exists x\ Q(x,y)$	N
$\forall x\ P(x) \vee \forall x\ Q(x,x)$	

	Formula?

Exercise

	Closed?
$\forall x\ P(a)$	Y
$\forall x \exists y\ (Q(x,y) \vee R(x,y))$	Y
$\forall x\ Q(x,x) \to \exists x\ Q(x,y)$	N
$\forall x\ P(x) \vee \forall x\ Q(x,x)$	Y

	Formula?

Exercise

	Closed?
$\forall x\ P(a)$	Y
$\forall x \exists y\ (Q(x,y) \vee R(x,y))$	Y
$\forall x\ Q(x,x) \rightarrow \exists x\ Q(x,y)$	N
$\forall x\ P(x) \vee \forall x\ Q(x,x)$	Y
$\forall x\ (P(y) \wedge \forall y\ P(x))$	

	Formula?

Exercise

	Closed?
$\forall x\ P(a)$	Y
$\forall x \exists y\ (Q(x,y) \vee R(x,y))$	Y
$\forall x\ Q(x,x) \to \exists x\ Q(x,y)$	N
$\forall x\ P(x) \vee \forall x\ Q(x,x)$	Y
$\forall x\ (P(y) \wedge \forall y\ P(x))$	N

	Formula?

Exercise

	Closed?
$\forall x\ P(a)$	Y
$\forall x \exists y\ (Q(x,y) \vee R(x,y))$	Y
$\forall x\ Q(x,x) \rightarrow \exists x\ Q(x,y)$	N
$\forall x\ P(x) \vee \forall x\ Q(x,x)$	Y
$\forall x\ (P(y) \wedge \forall y\ P(x))$	N
$P(x) \rightarrow \exists x\ Q(x,f(x))$	

	Formula?

Exercise

	Closed?
$\forall x\ P(a)$	Y
$\forall x \exists y\ (Q(x,y) \vee R(x,y))$	Y
$\forall x\ Q(x,x) \rightarrow \exists x\ Q(x,y)$	N
$\forall x\ P(x) \vee \forall x\ Q(x,x)$	Y
$\forall x\ (P(y) \wedge \forall y\ P(x))$	N
$P(x) \rightarrow \exists x\ Q(x, f(x))$	N

	Formula?

Exercise

	Closed?
$\forall x\ P(a)$	Y
$\forall x \exists y\ (Q(x,y) \vee R(x,y))$	Y
$\forall x\ Q(x,x) \rightarrow \exists x\ Q(x,y)$	N
$\forall x\ P(x) \vee \forall x\ Q(x,x)$	Y
$\forall x\ (P(y) \wedge \forall y\ P(x))$	N
$P(x) \rightarrow \exists x\ Q(x, f(x))$	N

	Formula?
$\exists x\ P(f(x))$	

Exercise

	Closed?
$\forall x\ P(a)$	Y
$\forall x \exists y\ (Q(x,y) \vee R(x,y))$	Y
$\forall x\ Q(x,x) \rightarrow \exists x\ Q(x,y)$	N
$\forall x\ P(x) \vee \forall x\ Q(x,x)$	Y
$\forall x\ (P(y) \wedge \forall y\ P(x))$	N
$P(x) \rightarrow \exists x\ Q(x, f(x))$	N

	Formula?
$\exists x\ P(f(x))$	Y

Exercise

	Closed?
$\forall x\ P(a)$	Y
$\forall x \exists y\ (Q(x,y) \vee R(x,y))$	Y
$\forall x\ Q(x,x) \rightarrow \exists x\ Q(x,y)$	N
$\forall x\ P(x) \vee \forall x\ Q(x,x)$	Y
$\forall x\ (P(y) \wedge \forall y\ P(x))$	N
$P(x) \rightarrow \exists x\ Q(x, f(x))$	N

	Formula?
$\exists x\ P(f(x))$	Y
$\exists f\ P(f(x))$	

Exercise

	Closed?
$\forall x\ P(a)$	Y
$\forall x \exists y\ (Q(x,y) \vee R(x,y))$	Y
$\forall x\ Q(x,x) \rightarrow \exists x\ Q(x,y)$	N
$\forall x\ P(x) \vee \forall x\ Q(x,x)$	Y
$\forall x\ (P(y) \wedge \forall y\ P(x))$	N
$P(x) \rightarrow \exists x\ Q(x,f(x))$	N

	Formula?
$\exists x\ P(f(x))$	Y
$\exists f\ P(f(x))$	N

Semantics of predicate logic: structures

A structure is a pair $\mathcal{A} = (U_{\mathcal{A}}, I_{\mathcal{A}})$
where $U_{\mathcal{A}}$ is an arbitrary, nonempty set called the universe of $\mathcal{A}$,
and the interpretation $I_{\mathcal{A}}$ is a partial function that maps

- variables to elements of the universe $U_{\mathcal{A}}$,
- function symbols of arity k to functions of type $U_{\mathcal{A}}^k \to U_{\mathcal{A}}$,
- predicate symbols of arity k to functions of type $U_{\mathcal{A}}^k \to \{0, 1\}$ (predicates) [or equivalently to subsets of $U_{\mathcal{A}}^k$ (relations)]

Semantics of predicate logic: structures

A structure is a pair $\mathcal{A} = (U_{\mathcal{A}}, I_{\mathcal{A}})$
where $U_{\mathcal{A}}$ is an arbitrary, nonempty set called the universe of $\mathcal{A}$,
and the interpretation $I_{\mathcal{A}}$ is a partial function that maps

- variables to elements of the universe $U_{\mathcal{A}}$,
- function symbols of arity k to functions of type $U_{\mathcal{A}}^k \to U_{\mathcal{A}}$,
- predicate symbols of arity k to functions of type $U_{\mathcal{A}}^k \to \{0, 1\}$ (predicates) [or equivalently to subsets of $U_{\mathcal{A}}^k$ (relations)]

$I_{\mathcal{A}}$ maps syntax (variables, functions and predicate symbols) to their meaning (elements, functions and predicates)

Semantics of predicate logic: structures

A structure is a pair $\mathcal{A} = (U_\mathcal{A}, I_\mathcal{A})$
where $U_\mathcal{A}$ is an arbitrary, nonempty set called the universe of $\mathcal{A}$,
and the interpretation $I_\mathcal{A}$ is a partial function that maps

- variables to elements of the universe $U_\mathcal{A}$,
- function symbols of arity k to functions of type $U_\mathcal{A}^k \to U_\mathcal{A}$,
- predicate symbols of arity k to functions of type $U_\mathcal{A}^k \to \{0, 1\}$ (predicates) [or equivalently to subsets of $U_\mathcal{A}^k$ (relations)]

$I_\mathcal{A}$ maps syntax (variables, functions and predicate symbols)
to their meaning (elements, functions and predicates)

The special case of arity 0 can be written more simply:

- constant symbols are mapped to elements of $U_\mathcal{A}$,
- predicate symbols of arity 0 are mapped to $\{0, 1\}$.

Abbreviations:

$$\begin{array}{lll} x^{\mathcal{A}} & \text{abbreviates} & I_{\mathcal{A}}(x) \\ f^{\mathcal{A}} & \text{abbreviates} & I_{\mathcal{A}}(f) \\ P^{\mathcal{A}} & \text{abbreviates} & I_{\mathcal{A}}(P) \end{array}$$

Abbreviations:

$$\begin{array}{lll} x^{\mathcal{A}} & \text{abbreviates} & I_{\mathcal{A}}(x) \\ f^{\mathcal{A}} & \text{abbreviates} & I_{\mathcal{A}}(f) \\ P^{\mathcal{A}} & \text{abbreviates} & I_{\mathcal{A}}(P) \end{array}$$

Example

$U_{\mathcal{A}} = \mathbb{N}$

$I_{\mathcal{A}}(P) = P^{\mathcal{A}} = \{(m, n) \mid m, n \in \mathbb{N} \text{ and } m < n\}$

$I_{\mathcal{A}}(Q) = Q^{\mathcal{A}} = \{m \mid m \in \mathbb{N} \text{ and } m \text{ is prime}\}$

$I_{\mathcal{A}}(f)$ is the successor function: $f^{\mathcal{A}}(n) = n + 1$

$I_{\mathcal{A}}(g)$ is the addition function: $g^{\mathcal{A}}(m, n) = m + n$

$I_{\mathcal{A}}(a) = a^{\mathcal{A}} = 2$

$I_{\mathcal{A}}(z) = z^{\mathcal{A}} = 3$

Intuition: is $\forall x\ P(x, f(x)) \wedge Q(g(a, z))$ true in this structure?

Evaluation of a term in a structure

Definition

Let t be a term and let $\mathcal{A} = (U_{\mathcal{A}}, I_{\mathcal{A}})$ be a structure.
$\mathcal{A}$ is suitable for t if $I_{\mathcal{A}}$ is defined for all variables and function symbols occurring in t.

Evaluation of a term in a structure

Definition

Let t be a term and let $\mathcal{A} = (U_\mathcal{A}, I_\mathcal{A})$ be a structure.
$\mathcal{A}$ is suitable for t if $I_\mathcal{A}$ is defined for all variables and function symbols occurring in t. The value of a term t in a suitable structure $\mathcal{A}$, denoted by $\mathcal{A}(t)$, is defined recursively:

$$
\begin{aligned}
\mathcal{A}(x) &= x^\mathcal{A} \\
\mathcal{A}(c) &= c^\mathcal{A} \\
\mathcal{A}(f(t_1, \ldots, t_k)) &= f^\mathcal{A}(\mathcal{A}(t_1), \ldots, \mathcal{A}(t_k))
\end{aligned}
$$

Definition

Let F be a formula and let $\mathcal{A} = (U_\mathcal{A}, I_\mathcal{A})$ be a structure.
$\mathcal{A}$ is suitable for F if $I_\mathcal{A}$ is defined for all predicate and function symbols occurring in F and for all variables occurring free in F.

Evaluation of a formula in a structure

Suitable structures for $\forall x(P(x) \to \exists y Q(x,y))$

$$\begin{aligned}
\mathcal{A}_1 &= (U_1, I_1) \\
U_1 &= \mathbb{N} \\
P^{\mathcal{A}_1} &= 2\mathbb{N} \\
Q^{\mathcal{A}_1} &= \{(n,m) \mid n+m=5\}
\end{aligned}$$

$$\begin{aligned}
\mathcal{A}_3 &= (U_3, I_3) \\
U_3 &= \{a, b\} \\
P^{\mathcal{A}_3} &= \{a\} \\
Q^{\mathcal{A}_3} &= \{(a,b),(b,a),(b,b)\}
\end{aligned}$$

$$\begin{aligned}
\mathcal{A}_2 &= (U_2, I_2) \\
U_2 &= \{0,1,2\} \\
P^{\mathcal{A}_2} &= \{0\} \\
Q^{\mathcal{A}_2} &= \{(n,m) \mid n \leq m\}
\end{aligned}$$

$$\begin{aligned}
\mathcal{A}_4 &= (U_4, I_4) \\
U_4 &= \mathbb{N} \\
P^{\mathcal{A}_4} &= \emptyset \\
Q^{\mathcal{A}_4} &= \emptyset
\end{aligned}$$

Evaluation of a formula in a structure

Let $\mathcal{A}$ be suitable for F. The (truth) value of F in $\mathcal{A}$, denoted by $\mathcal{A}(F)$, is defined recursively:

$$\mathcal{A}(\neg F),\ \mathcal{A}(F \wedge G),\ \mathcal{A}(F \vee G),\ \mathcal{A}(F \rightarrow G)$$

as for propositional logic.

Evaluation of a formula in a structure

Let $\mathcal{A}$ be suitable for F. The (truth) value of F in $\mathcal{A}$, denoted by $\mathcal{A}(F)$, is defined recursively:

$$\mathcal{A}(\neg F),\ \mathcal{A}(F \wedge G),\ \mathcal{A}(F \vee G),\ \mathcal{A}(F \rightarrow G)$$

as for propositional logic.

$$\mathcal{A}(P(t_1, \ldots, t_k)) = \begin{cases} 1 & \text{if } (\mathcal{A}(t_1), \ldots, \mathcal{A}(t_k)) \in P^{\mathcal{A}} \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{A}(\forall x\ F) = \begin{cases} 1 & \text{if for every } d \in U_{\mathcal{A}},\ (\mathcal{A}[d/x])(F) = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{A}(\exists x\ F) = \begin{cases} 1 & \text{if for some } d \in U_{\mathcal{A}},\ (\mathcal{A}[d/x])(F) = 1 \\ 0 & \text{otherwise} \end{cases}$$

Recall: $\mathcal{A}[d/x]$ coincides with $\mathcal{A}$ except $x^{\mathcal{A}[d/x]} = d$.

Notes

- During the evaluation of a formulas in a structure, the structure stays unchanged except for the interpretation of the variables.
- If the formula is closed, the initial interpretation of the variables is irrelevant.

Coincidence Lemma

Lemma

Let $\mathcal{A}$ and $\mathcal{A}'$ be two structures that coincide on all free variables, on all function symbols and all predicate symbols that occur in F. Then $\mathcal{A}(F) = \mathcal{A}'(F)$.

Proof.

Exercise. $\square$

Relation to propositional logic

- Every propositional formula can be seen as a formula of predicate logic where the atom A_i is replaced by the atom P_i^0.

Relation to propositional logic

- Every propositional formula can be seen as a formula of predicate logic where the atom A_i is replaced by the atom P_i^0.
- Conversely, every formula of predicate logic that does not contain quantifiers and variables can be seen as a formula of propositional logic by replacing atomic formulas by propositional atoms.

Relation to propositional logic

- Every propositional formula can be seen as a formula of predicate logic where the atom A_i is replaced by the atom P_i^0.
- Conversely, every formula of predicate logic that does not contain quantifiers and variables can be seen as a formula of propositional logic by replacing atomic formulas by propositional atoms.

Example

$$F = Q(a) \vee \neg P(f(b), b) \wedge P(b, f(b))$$

can be viewed as the propositional formula

$$F' = A_1 \vee \neg A_2 \wedge A_3 \ .$$

Relation to propositional logic

- Every propositional formula can be seen as a formula of predicate logic where the atom A_i is replaced by the atom P_i^0.
- Conversely, every formula of predicate logic that does not contain quantifiers and variables can be seen as a formula of propositional logic by replacing atomic formulas by propositional atoms.

Example

$$F = Q(a) \vee \neg P(f(b), b) \wedge P(b, f(b))$$

can be viewed as the propositional formula

$$F' = A_1 \vee \neg A_2 \wedge A_3 \ .$$

Predicate logic with equality

Predicate logic

$+$

distinguished predicate symbol "=" of arity 2

Semantics: A structure $\mathcal{A}$ of predicate logic with equality always maps the predicate symbol = to the identity relation:

$$\mathcal{A}(=) \; = \; \{(d, d) \mid d \in U_{\mathcal{A}}\}$$

Formalizing statements

"Formalizing" a statement in predicate logic means to give a formula F **and** a structure $\mathcal{A}$ such that "the meaning of F in $\mathcal{A}$ corresponds to the statement." More prcisely: every structure that extends $\mathcal{A}$ is a model of F.

Formalizing statements

"Formalizing" a statement in predicate logic means to give a formula F **and** a structure $\mathcal{A}$ such that "the meaning of F in $\mathcal{A}$ corresponds to the statement." More prcisely: every structure that extends $\mathcal{A}$ is a model of F.

Intuitively, the symbols interpreted in $\mathcal{A}$ are those that the formalizer assumes are known by whoever is going to read the formula. F may contain other symbols, but then F must define what they mean (see next slides).

Formalizing statements

"Formalizing" a statement in predicate logic means to give a formula F **and** a structure $\mathcal{A}$ such that "the meaning of F in $\mathcal{A}$ corresponds to the statement." More prcisely: every structure that extends $\mathcal{A}$ is a model of F.

Intuitively, the symbols interpreted in $\mathcal{A}$ are those that the formalizer assumes are known by whoever is going to read the formula. F may contain other symbols, but then F must define what they mean (see next slides).

Typically, the formalizer chooses names for the symbols that suggest their meaning. The structure is often omitted, because it is assumed to be known (danger!).

Formalizing statements

"Formalizing" a statement in predicate logic means to give a formula F **and** a structure $\mathcal{A}$ such that "the meaning of F in $\mathcal{A}$ corresponds to the statement." More prcisely: every structure that extends $\mathcal{A}$ is a model of F.

Intuitively, the symbols interpreted in $\mathcal{A}$ are those that the formalizer assumes are known by whoever is going to read the formula. F may contain other symbols, but then F must define what they mean (see next slides).

Typically, the formalizer chooses names for the symbols that suggest their meaning. The structure is often omitted, because it is assumed to be known (danger!).

We give different formalizations of the statement

There are infinitely many prime numbers

Formalization I

If the meanings of “prime” and “‘greater-than” are known, then we can take:

F_1: $\forall x \, \exists y \, (Pr(y) \wedge y > x)$

$$\mathcal{A}_1: \quad \begin{array}{rcl} U^{\mathcal{A}_1} & = & \mathbb{N} \\ Pr^{\mathcal{A}_1} & = & \{n \in \mathbb{N} \mid n \text{ is prime}\} \\ >^{\mathcal{A}_1} & = & \{(n, m) \in \mathbb{N} \mid n > m\} \end{array}$$

Formalization I

If the meanings of “prime” and “‘greater-than” are known, then we can take:

F_1: $\forall x \, \exists y \, (Pr(y) \land y > x)$

$$
\mathcal{A}_1: \quad \begin{array}{rcl}
U^{\mathcal{A}_1} & = & \mathbb{N} \\
Pr^{\mathcal{A}_1} & = & \{n \in \mathbb{N} \mid n \text{ is prime}\} \\
>^{\mathcal{A}_1} & = & \{(n, m) \in \mathbb{N} \mid n > m\}
\end{array}
$$

What if the meaning of “prime” is not known?

Formalization II

If the meaning of "divides" and "one" are known, then we can take:

$$F_2: \quad \forall x\,(Pr(x) \leftrightarrow \forall y\,(Dv(y,x) \rightarrow (y = x \vee y = one))) \rightarrow \forall x\,\exists y\,(Pr(y) \wedge y > x)$$

$$\mathcal{A}_2: \quad \begin{aligned} U^{\mathcal{A}_2} &= \mathbb{N} \\ Dv^{\mathcal{A}_2} &= \{(n,m) \in \mathbb{N} \mid n \text{ divides } m\} \\ >^{\mathcal{A}_2} &= \{(n,m) \in \mathbb{N} \mid n > m\} \\ one^{\mathcal{A}_2} &= 1 \end{aligned}$$

We are now stating " if we define prime numbers as ... then there are infinitely many prime numbers".

Formalization II

If the meaning of "divides" and "one" are known, then we can take:

$$
\begin{array}{rl}
F_2: & \forall x\,(Pr(x) \leftrightarrow \forall y\,(Dv(y,x) \rightarrow (y = x \vee y = one))) \\
\rightarrow & \forall x\,\exists y\,(Pr(y) \wedge y > x)
\end{array}
$$

$$
\mathcal{A}_2: \quad
\begin{array}{rcl}
U^{\mathcal{A}_2} & = & \mathbb{N} \\
Dv^{\mathcal{A}_2} & = & \{(n,m) \in \mathbb{N} \mid n \text{ divides } m\} \\
>^{\mathcal{A}_2} & = & \{(n,m) \in \mathbb{N} \mid n > m\} \\
one^{\mathcal{A}_2} & = & 1
\end{array}
$$

We are now stating " if we define prime numbers as ... then there are infinitely many prime numbers".

What if the meaning of "divides" is not known?

Formalization III

If the meaning of "product" is known , then we can take

$$
\begin{array}{lrl}
F_3: & & \forall x\, \forall y\, (Dv(x,y) \leftrightarrow \exists z\, prod(x,z) = y) \\
& \wedge & \forall x\, (Pr(x) \leftrightarrow (\forall y\, Dv(y,x) \rightarrow (y = x \vee y = one))) \\
& \rightarrow & \forall x\, \exists y\, (Pr(y) \wedge y > x)
\end{array}
$$

(the conjunction of the first two formulas implies the third)

$$
\begin{array}{lrcl}
\mathcal{A}_3: & U^{\mathcal{A}_3} & = & \mathbb{N} \\
& >^{\mathcal{A}_3} & = & \{(n,m) \in \mathbb{N} \mid n > m\} \\
& one^{\mathcal{A}_3} & = & 1 \\
& prod^{\mathcal{A}_3}(n,m) & = & n \cdot m
\end{array}
$$

Formalization III

If the meaning of "product" is known , then we can take

$$
\begin{array}{lll}
F_3\colon & & \forall x\, \forall y\, (Dv(x,y) \leftrightarrow \exists z\, prod(x,z) = y) \\
& \wedge & \forall x\, (Pr(x) \leftrightarrow (\forall y\, Dv(y,x) \rightarrow (y = x \vee y = one))) \\
& \rightarrow & \forall x\, \exists y\, (Pr(y) \wedge y > x)
\end{array}
$$

(the conjunction of the first two formulas implies the third)

$$
\begin{array}{lrcl}
\mathcal{A}_3\colon & U^{\mathcal{A}_3} & = & \mathbb{N} \\
& >^{\mathcal{A}_3} & = & \{(n,m) \in \mathbb{N} \mid n > m\} \\
& one^{\mathcal{A}_3} & = & 1 \\
& prod^{\mathcal{A}_3}(n,m) & = & n \cdot m
\end{array}
$$

What if the meaning of "product" is not known ?

Formalization IV

If the meaning of "sum", "successor", "one" and "zero" is known, then we can take

$$
\begin{array}{ll}
F_4: & \forall x \; prod(x, zero) = zero \\
\wedge & \forall x \, \forall y \, prod(x, succ(y)) = sum(prod(x, y), y) \\
\wedge & \forall x \, \forall y \, (Dv(x, y) \leftrightarrow \exists z \, prod(x, z) = y) \\
\wedge & \forall x \, (Pr(x) \leftrightarrow (\forall y \, Dv(y, x) \rightarrow (y = x \vee y = one))) \\
\rightarrow & \forall x \, \exists y \; (Pr(y) \wedge y > x)
\end{array}
$$

$\mathcal{A}_4$ only defines $>, sum, succ, one, zero$.

Observe: *prod* is defined *inductively*. The definition is no longer a macro, in the sense that we cannot produce an "equivalent" formula without the symbol *prod*.

Formalization IV

If the meaning of "sum", "successor", "one" and "zero" is known, then we can take

$$
\begin{array}{lll}
F_4: & & \forall x \; prod(x, zero) = zero \\
& \wedge & \forall x \, \forall y \, prod(x, succ(y)) = sum(prod(x, y), y) \\
& \wedge & \forall x \, \forall y \, (Dv(x, y) \leftrightarrow \exists z \, prod(x, z) = y) \\
& \wedge & \forall x \, (Pr(x) \leftrightarrow (\forall y \, Dv(y, x) \rightarrow (y = x \vee y = one))) \\
& \rightarrow & \forall x \, \exists y \; (Pr(y) \wedge y > x)
\end{array}
$$

$\mathcal{A}_4$ only defines $>, sum, succ, one, zero$.

Observe: *prod* is defined *inductively*. The definition is no longer a macro, in the sense that we cannot produce an "equivalent" formula without the symbol *prod*.

What if the meaning of "sum" is not known?

Formalization V

$$
\begin{array}{lll}
F_5: & & \forall x\, sum(x, zero) = x \\
& \wedge & \forall x\, \forall y\, sum(x, succ(y)) = succ(sum(x, y)) \\
& \wedge & \forall x\, prod(x, zero) = zero \\
& \wedge & \forall x\, \forall y\, prod(x, succ(y)) = sum(prod(x, y), y) \\
& \wedge & \forall x\, \forall y\, (Div(x, y) \leftrightarrow \exists z\, prod(x, z) = y) \\
& \wedge & \forall x\, (Pr(x) \leftrightarrow (\forall y\; Div(y, x) \rightarrow (y = x \vee y = one))) \\
& \rightarrow & \forall x\, \exists y\, (Pr(y) \wedge y > x)
\end{array}
$$

$\mathcal{A}_5$ only defines $>, succ, one, zero$.

Formalization V

$$
\begin{array}{lll}
F_5: & & \forall x\, sum(x, zero) = x \\
& \wedge & \forall x\, \forall y\, sum(x, succ(y)) = succ(sum(x, y)) \\
& \wedge & \forall x\, prod(x, zero) = zero \\
& \wedge & \forall x\, \forall y\, prod(x, succ(y)) = sum(prod(x, y), y) \\
& \wedge & \forall x\, \forall y\, (Div(x, y) \leftrightarrow \exists z\, prod(x, z) = y) \\
& \wedge & \forall x\, (Pr(x) \leftrightarrow (\forall y\;\; Div(y, x) \rightarrow (y = x \vee y = one))) \\
& \rightarrow & \forall x\, \exists y\, (Pr(y) \wedge y > x)
\end{array}
$$

$\mathcal{A}_5$ only defines $>, succ, one, zero$.

What if the meaning of ‘greater than” and “one” is not known?

Formalization VI

$$
\begin{array}{lll}
F_6: & & one = succ(zero) \\
 & \wedge & \forall x \, \forall y \, (x > y \leftrightarrow \exists z \, \neg(z = zero) \wedge sum(y, z) = x) \\
 & \wedge & \forall x \, sum(x, zero) = x \\
 & \wedge & \forall x \, \forall y \, sum(x, succ(y)) = succ(sum(x, y)) \\
 & \wedge & \forall x \, prod(x, zero) = zero \\
 & \wedge & \forall x \, \forall y \, prod(x, succ(y)) = sum(prod(x, y), y) \\
 & \wedge & \forall x \, \forall y \, (Div(x, y) \leftrightarrow \exists z \;\, prod(x, z) = y) \\
 & \wedge & \forall x \, (Pr(x) \leftrightarrow (\forall y \;\, Div(y, x) \rightarrow (y = x \vee y = one))) \\
 & \rightarrow & \forall x \, \exists y \, (Pr(y) \wedge y > x)
\end{array}
$$

$\mathcal{A}_6$ only defines $succ, zero$.

Model, validity, satisfiability

Like in propositional logic

Definition

We write $\mathcal{A} \models F$ to denote that the structure $\mathcal{A}$ is suitable for the formula F and that $\mathcal{A}(F) = 1$.
Then we say that F is true in $\mathcal{A}$ or that $\mathcal{A}$ is a model of F.

If every structure suitable for F is a model of F,
then we write $\models F$ and say that F is valid.

If F has at least one model then we say that F is satisfiable.

Exercise

V: valid   S: satisfiable, but not valid   U: unsatisfiable

	V	S	U
$\forall x\, P(a)$			

Exercise

V: valid   S: satisfiable, but not valid   U: unsatisfiable

	V	S	U
$\forall x\, P(a)$		x	

Exercise

V: valid   S: satisfiable, but not valid   U: unsatisfiable

	V	S	U
$\forall x\, P(a)$		x	
$\exists x\, (\neg P(x) \vee P(a))$			

Exercise

V: valid   S: satisfiable, but not valid   U: unsatisfiable

	V	S	U
$\forall x\, P(a)$		x	
$\exists x\, (\neg P(x) \vee P(a))$	x		

Exercise

V: valid S: satisfiable, but not valid U: unsatisfiable

	V	S	U
$\forall x\, P(a)$		x	
$\exists x\, (\neg P(x) \vee P(a))$	x		
$P(a) \to \exists x\, P(x)$			

Exercise

V: valid   S: satisfiable, but not valid   U: unsatisfiable

	V	S	U
$\forall x\, P(a)$		x	
$\exists x\, (\neg P(x) \vee P(a))$	x		
$P(a) \rightarrow \exists x\, P(x)$	x		

Exercise

V: valid S: satisfiable, but not valid U: unsatisfiable

	V	S	U
$\forall x\, P(a)$		x	
$\exists x\, (\neg P(x) \vee P(a))$	x		
$P(a) \rightarrow \exists x\, P(x)$	x		
$P(x) \rightarrow \exists x\, P(x)$			

Exercise

V: valid S: satisfiable, but not valid U: unsatisfiable

	V	S	U
$\forall x\, P(a)$		x	
$\exists x\, (\neg P(x) \vee P(a))$	x		
$P(a) \to \exists x\, P(x)$	x		
$P(x) \to \exists x\, P(x)$	x		

Exercise

V: valid   S: satisfiable, but not valid   U: unsatisfiable

	V	S	U
$\forall x\, P(a)$		x	
$\exists x\, (\neg P(x) \vee P(a))$	x		
$P(a) \to \exists x\, P(x)$	x		
$P(x) \to \exists x\, P(x)$	x		
$\forall x\, P(x) \to \exists x\, P(x)$			

Exercise

V: valid S: satisfiable, but not valid U: unsatisfiable

	V	S	U
$\forall x \, P(a)$		x	
$\exists x \, (\neg P(x) \vee P(a))$	x		
$P(a) \rightarrow \exists x \, P(x)$	x		
$P(x) \rightarrow \exists x \, P(x)$	x		
$\forall x \, P(x) \rightarrow \exists x \, P(x)$	x		

Exercise

V: valid   S: satisfiable, but not valid   U: unsatisfiable

	V	S	U
$\forall x\, P(a)$		x	
$\exists x\, (\neg P(x) \vee P(a))$	x		
$P(a) \rightarrow \exists x\, P(x)$	x		
$P(x) \rightarrow \exists x\, P(x)$	x		
$\forall x\, P(x) \rightarrow \exists x\, P(x)$	x		
$\forall x\, P(x) \wedge \neg \forall y\, P(y)$			

Exercise

V: valid S: satisfiable, but not valid U: unsatisfiable

	V	S	U
$\forall x\, P(a)$		x	
$\exists x\, (\neg P(x) \vee P(a))$	x		
$P(a) \rightarrow \exists x\, P(x)$	x		
$P(x) \rightarrow \exists x\, P(x)$	x		
$\forall x\, P(x) \rightarrow \exists x\, P(x)$	x		
$\forall x\, P(x) \wedge \neg \forall y\, P(y)$			x

Consequence and equivalence

Like in propositional logic

Definition

A formula G is a consequence of a set of formulas M
if every structure that is a model of all $F \in M$ and suitable for G
is also a model of G. Then we write $M \models G$.

Two formulas F and G are (semantically) equivalent
if every structure $\mathcal{A}$ suitable for both F and G satisfies
$\mathcal{A}(F) = \mathcal{A}(G)$. Then we write $F \equiv G$.

Exercise

1. $\forall x\, P(x) \vee \forall x\, Q(x,x)$
2. $\forall x\, (P(x) \vee Q(x,x))$
3. $\forall x\, (\forall z\, P(z) \vee \forall y\, Q(x,y))$

	Y	N
$1 \models 2$		

Exercise

1. $\forall x\, P(x) \lor \forall x\, Q(x, x)$
2. $\forall x\, (P(x) \lor Q(x, x))$
3. $\forall x\, (\forall z\, P(z) \lor \forall y\, Q(x, y))$

	Y	N
$1 \models 2$	x	

Exercise

1. $\forall x\, P(x) \vee \forall x\, Q(x,x)$
2. $\forall x\, (P(x) \vee Q(x,x))$
3. $\forall x\, (\forall z\, P(z) \vee \forall y\, Q(x,y))$

	Y	N
$1 \models 2$	x	
$2 \models 3$		

Exercise

1. $\forall x\, P(x) \vee \forall x\, Q(x,x)$
2. $\forall x\, (P(x) \vee Q(x,x))$
3. $\forall x\, (\forall z\, P(z) \vee \forall y\, Q(x,y))$

	Y	N
$1 \models 2$	x	
$2 \models 3$		x

Exercise

1. $\forall x\, P(x) \vee \forall x\, Q(x,x)$
2. $\forall x\, (P(x) \vee Q(x,x))$
3. $\forall x\, (\forall z\, P(z) \vee \forall y\, Q(x,y))$

	Y	N
$1 \models 2$	x	
$2 \models 3$		x
$3 \models 1$		

Exercise

1. $\forall x\, P(x) \vee \forall x\, Q(x, x)$
2. $\forall x\, (P(x) \vee Q(x, x))$
3. $\forall x\, (\forall z\, P(z) \vee \forall y\, Q(x, y))$

	Y	N
$1 \models 2$	x	
$2 \models 3$		x
$3 \models 1$	x	

Exercise

1. $\exists y \, \forall x \, P(x, y)$
2. $\forall x \, \exists y \, P(x, y)$

	Y	N
$1 \models 2$		

Exercise

1. $\exists y\, \forall x\, P(x, y)$
2. $\forall x\, \exists y\, P(x, y)$

	Y	N
$1 \models 2$	x	

Exercise

1. $\exists y \, \forall x \, P(x, y)$
2. $\forall x \, \exists y \, P(x, y)$

	Y	N
$1 \models 2$	x	
$2 \models 1$		

Exercise

1. $\exists y \, \forall x \, P(x, y)$
2. $\forall x \, \exists y \, P(x, y)$

	Y	N
$1 \models 2$	x	
$2 \models 1$		x

Exercise

	Y	N
$\forall x \forall y\, F \equiv \forall y \forall x\, F$		

Exercise

	Y	N
$\forall x \forall y\, F \equiv \forall y \forall x\; F$	x	

Exercise

	Y	N
$\forall x \, \forall y \, F \; \equiv \; \forall y \, \forall x \; F$	x	
$\forall x \, \exists y \, F \; \equiv \; \exists x \, \forall y \, F$		

Exercise

	Y	N
$\forall x \forall y F \equiv \forall y \forall x \; F$	x	
$\forall x \exists y F \equiv \exists x \forall y F$		x

Exercise

	Y	N
$\forall x \, \forall y \, F \;\equiv\; \forall y \, \forall x \;\, F$	x	
$\forall x \, \exists y \, F \;\equiv\; \exists x \, \forall y \, F$		x
$\exists x \, \exists y \;\, F \;\equiv\; \exists y \, \exists x \, F$		

Exercise

	Y	N
$\forall x\,\forall y\,F \;\equiv\; \forall y\,\forall x\;F$	x	
$\forall x\,\exists y\,F \;\equiv\; \exists x\,\forall y\,F$		x
$\exists x\,\exists y\;F \;\equiv\; \exists y\,\exists x\,F$	x	

Exercise

	Y	N
$\forall x\, \forall y\, F \;\equiv\; \forall y\, \forall x\; F$	x	
$\forall x\, \exists y\, F \;\equiv\; \exists x\, \forall y\, F$		x
$\exists x\, \exists y\; F \;\equiv\; \exists y\, \exists x\, F$	x	
$\forall x\, F \vee \forall x\, G \;\equiv\; \forall x\, (F \vee G)$		

Exercise

	Y	N
$\forall x \, \forall y \, F \;\equiv\; \forall y \, \forall x \; F$	x	
$\forall x \, \exists y \, F \;\equiv\; \exists x \, \forall y \, F$		x
$\exists x \, \exists y \; F \;\equiv\; \exists y \, \exists x \, F$	x	
$\forall x \, F \vee \forall x \, G \;\equiv\; \forall x \, (F \vee G)$		x

Exercise

	Y	N
$\forall x\, \forall y\, F \;\equiv\; \forall y\, \forall x\; F$	x	
$\forall x\, \exists y\, F \;\equiv\; \exists x\, \forall y\, F$		x
$\exists x\, \exists y\; F \;\equiv\; \exists y\, \exists x\, F$	x	
$\forall x\, F \vee \forall x\, G \;\equiv\; \forall x\, (F \vee G)$		x
$\forall x\, F \wedge \forall x\, G \;\equiv\; \forall x\, (F \wedge G)$		

Exercise

	Y	N
$\forall x\, \forall y\, F \;\equiv\; \forall y\, \forall x\; F$	x	
$\forall x\, \exists y\, F \;\equiv\; \exists x\, \forall y\, F$		x
$\exists x\, \exists y\; F \;\equiv\; \exists y\, \exists x\, F$	x	
$\forall x\, F \vee \forall x\, G \;\equiv\; \forall x\, (F \vee G)$		x
$\forall x\, F \wedge \forall x\, G \;\equiv\; \forall x\, (F \wedge G)$	x	

Exercise

	Y	N
$\forall x\, \forall y\, F \;\equiv\; \forall y\, \forall x\;\; F$	x	
$\forall x\, \exists y\, F \;\equiv\; \exists x\, \forall y\, F$		x
$\exists x\, \exists y\;\; F \;\equiv\; \exists y\, \exists x\, F$	x	
$\forall x\, F \vee \forall x\, G \;\equiv\; \forall x\, (F \vee G)$		x
$\forall x\, F \wedge \forall x\, G \;\equiv\; \forall x\, (F \wedge G)$	x	
$\exists x\, F \vee \exists x\, G \;\equiv\; \exists x\, (F \vee G)$		

Exercise

	Y	N
$\forall x\, \forall y\, F \;\equiv\; \forall y\, \forall x\; F$	x	
$\forall x\, \exists y\, F \;\equiv\; \exists x\, \forall y\, F$		x
$\exists x\, \exists y\; F \;\equiv\; \exists y\, \exists x\, F$	x	
$\forall x\, F \vee \forall x\, G \;\equiv\; \forall x\,(F \vee G)$		x
$\forall x\, F \wedge \forall x\, G \;\equiv\; \forall x\,(F \wedge G)$	x	
$\exists x\, F \vee \exists x\, G \;\equiv\; \exists x\,(F \vee G)$	x	

Exercise

	Y	N
$\forall x\, \forall y\, F \;\equiv\; \forall y\, \forall x\; F$	x	
$\forall x\, \exists y\, F \;\equiv\; \exists x\, \forall y\, F$		x
$\exists x\, \exists y\; F \;\equiv\; \exists y\, \exists x\, F$	x	
$\forall x\, F \vee \forall x\, G \;\equiv\; \forall x\, (F \vee G)$		x
$\forall x\, F \wedge \forall x\, G \;\equiv\; \forall x\, (F \wedge G)$	x	
$\exists x\, F \vee \exists x\, G \;\equiv\; \exists x\, (F \vee G)$	x	
$\exists x\, F \wedge \exists x\, G \;\equiv\; \exists x\, (F \wedge G)$		

Exercise

	Y	N
$\forall x\, \forall y\, F \;\equiv\; \forall y\, \forall x\;\; F$	x	
$\forall x\, \exists y\, F \;\equiv\; \exists x\, \forall y\, F$		x
$\exists x\, \exists y\;\; F \;\equiv\; \exists y\, \exists x\, F$	x	
$\forall x\, F \vee \forall x\, G \;\equiv\; \forall x\, (F \vee G)$		x
$\forall x\, F \wedge \forall x\, G \;\equiv\; \forall x\, (F \wedge G)$	x	
$\exists x\, F \vee \exists x\, G \;\equiv\; \exists x\, (F \vee G)$	x	
$\exists x\, F \wedge \exists x\, G \;\equiv\; \exists x\, (F \wedge G)$		x

Exercise

	Y	N
$P(x) \equiv \exists x\, P(x)$		

Exercise

	Y	N
$P(x) \equiv \exists x\, P(x)$		x

Exercise

	Y	N
$P(x) \equiv \exists x\, P(x)$		x
$P(x) \equiv \forall x\, P(x)$		

Exercise

	Y	N
$P(x) \equiv \exists x\, P(x)$		x
$P(x) \equiv \forall x\, P(x)$		x

Exercise

	Y	N
$P(x) \equiv \exists x\, P(x)$		x
$P(x) \equiv \forall x\, P(x)$		x
$P(a) \equiv P(x)$		

Exercise

	Y	N
$P(x) \equiv \exists x\, P(x)$		x
$P(x) \equiv \forall x\, P(x)$		x
$P(a) \equiv P(x)$		x

Exercise

	Y	N
$P(x) \equiv \exists x\, P(x)$		x
$P(x) \equiv \forall x\, P(x)$		x
$P(a) \equiv P(x)$		x
$P(x) \equiv P(y)$		

Exercise

	Y	N
$P(x) \equiv \exists x\, P(x)$		x
$P(x) \equiv \forall x\, P(x)$		x
$P(a) \equiv P(x)$		x
$P(x) \equiv P(y)$		x

Exercise

	Y	N
$P(x) \equiv \exists x\, P(x)$		x
$P(x) \equiv \forall x\, P(x)$		x
$P(a) \equiv P(x)$		x
$P(x) \equiv P(y)$		x
$\forall x\, \forall y\, P(y) \equiv \forall x\, P(x)$		

Exercise

	Y	N
$P(x) \equiv \exists x\, P(x)$		x
$P(x) \equiv \forall x\, P(x)$		x
$P(a) \equiv P(x)$		x
$P(x) \equiv P(y)$		x
$\forall x\, \forall y\, P(y) \equiv \forall x\, P(x)$	x	

Exercise

	Y	N
$P(x) \equiv \exists x\, P(x)$		x
$P(x) \equiv \forall x\, P(x)$		x
$P(a) \equiv P(x)$		x
$P(x) \equiv P(y)$		x
$\forall x\, \forall y\, P(y) \equiv \forall x\, P(x)$	x	
$\exists x\, \forall y\, P(y) \equiv \forall x\, P(x)$		

Exercise

	Y	N
$P(x) \equiv \exists x\, P(x)$		x
$P(x) \equiv \forall x\, P(x)$		x
$P(a) \equiv P(x)$		x
$P(x) \equiv P(y)$		x
$\forall x\, \forall y\, P(y) \equiv \forall x\, P(x)$	x	
$\exists x\, \forall y\, P(y) \equiv \forall x\, P(x)$	x	

Equivalences

Theorem

1. $\neg\forall x\, F \equiv \exists x\, \neg F$
   $\neg\exists x\, F \equiv \forall x\, \neg F$
2. *If x does not occur free in G then:*
   $\forall x\, F \wedge G \equiv \forall x\, (F \wedge G)$
   $\forall x\, F \vee G \equiv \forall x\, (F \vee G)$
   $\exists x\, F \wedge G \equiv \exists x\, (F \wedge G)$
   $\exists x\, F \vee G \equiv \exists x\, (F \vee G)$
3. $\forall x\, F \wedge \forall x\, G \equiv \forall x\, (F \wedge G)$
   $\exists x F \vee \exists x G \equiv \exists x(F \vee G)$
4. $\forall x\, \forall y\, F \equiv \forall y\, \forall x\, F$
   $\exists x\, \exists y\, F \equiv \exists y\, \exists x\, F$

Replacement theorem

Just like for propositional logic it can be proved:

Theorem

Let $F \equiv G$. Let H be a formula with an occurrence of F as a subformula. Then $H \equiv H'$, where H' is the result of replacing an arbitrary occurrence of F in H by G.