Hilbert Systems
Propositional Logic

(See the book by Troelstra and Schwichtenberg)

Easy to define, hard to use.

No context management.

Hilbert systems and proof trees

A Hilbert system for propositional logic consists of

- a set of axioms axioms (formulas over formula variables)
- and a single inference rule, $\rightarrow E$ or modus ponens:

$$\frac{F \rightarrow G \quad F}{G} \quad \rightarrow E$$

Nodes of a proof tree are (labeled with) formulas.

Proof trees are defined inductively:

- Every formula F is a proof tree.
- Larger proof trees are constructed using $\rightarrow E$ (and $\rightarrow E$ only).

$\Gamma \vdash_H F$ denotes that there is a proof tree with root F whose leaves are either instances of axioms or elements of Γ (assumptions).

Alternative presentation

Proofs in Hilbert systems are frequently shown as lists of lines

$$\begin{array}{ll} 1\colon\ F_1 & \mathit{justification}_1 \\ 2\colon\ F_2 & \mathit{justification}_2 \\ \vdots & \\ i\colon\ F_i & \mathit{justification}_i \\ \vdots & \\ n\colon\ F_n & \mathit{justification}_n \end{array}$$

where *justification*$_i$ is either $\begin{cases} \mathit{assumption}, \\ \mathit{axiom}, \\ \rightarrow E\ (j,k), \text{ with } j,k < i. \end{cases}$

Notational convention:

$$F \to G \to H \quad \text{means} \quad F \to (G \to H)$$

Note: $F \to G \to H \quad \equiv \quad F \land G \to H$

$F \to G \to H \quad \not\equiv \quad (F \to G) \to H$

A simple Hilbert system

Axioms:

$$F \to G \to F \qquad \text{(A1)}$$

$$(F \to G \to H) \to (F \to G) \to F \to H \qquad \text{(A2)}$$

A proof of $A \to A$:

A simple Hilbert system

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A proof of $A \to A$:

1 :

2 :

3 :

4 :

5 : $A \to A$

A simple Hilbert system

Axioms: $F \to G \to F$ (A1)

$(F \to G \to H) \to (F \to G) \to F \to H$ (A2)

A proof of $A \to A$:

1 :

2 :

3 : $\_\_\_\_\_\_\_\_\_\_ \to A \to A$

4 : $\_\_\_\_\_\_\_\_\_\_$

5 : $A \to A$ $\to\! E : 3, 4$

A simple Hilbert system

Axioms: $F \to G \to F$ (A1)

$(F \to G \to H) \to (F \to G) \to F \to H$ (A2)

A proof of $A \to A$:

1 :

2 :

3 : $(A \to A \to A) \to A \to A$

4 : $A \to A \to A$

5 : $A \to A$ $\qquad \to E : 3, 4$

A simple Hilbert system

Axioms:
$$F \to G \to F \qquad \text{(A1)}$$
$$(F \to G \to H) \to (F \to G) \to F \to H \qquad \text{(A2)}$$

A proof of $A \to A$:

1 :

2 :

3 : $(A \to A \to A) \to A \to A$

4 : $A \to A \to A$ A1

5 : $A \to A$ $\to E : 3, 4$

A simple Hilbert system

Axioms: $F \to G \to F$ (A1)

$(F \to G \to H) \to (F \to G) \to F \to H$ (A2)

A proof of $A \to A$:

1 : ________________ $\to (A \to A \to A) \to A \to A$

2 : ________________

3 : $(A \to A \to A) \to A \to A$ $\to E$: 2, 1

4 : $A \to A \to A$ A1

5 : $A \to A$ $\to E$: 3, 4

A simple Hilbert system

Axioms: $F \to G \to F$ (A1)

$(F \to G \to H) \to (F \to G) \to F \to H$ (A2)

A proof of $A \to A$:

$1: (A \to (A \to A) \to A) \to (A \to A \to A) \to A \to A$

$2: A \to (A \to A) \to A$

$3: (A \to A \to A) \to A \to A$ $\quad \to E : 2, 1$

$4: A \to A \to A$ $\quad$ A1

$5: A \to A$ $\quad \to E : 3, 4$

A simple Hilbert system

Axioms:

$$F \to G \to F \tag{A1}$$

$$(F \to G \to H) \to (F \to G) \to F \to H \tag{A2}$$

A proof of $A \to A$:

$1 : (A \to (A \to A) \to A) \to (A \to A \to A) \to A \to A$

$2 : A \to (A \to A) \to A$ A1

$3 : (A \to A \to A) \to A \to A$ $\to E : 2, 1$

$4 : A \to A \to A$ A1

$5 : A \to A$ $\to E : 3, 4$

A simple Hilbert system

Axioms:

$$F \to G \to F \qquad \text{(A1)}$$

$$(F \to G \to H) \to (F \to G) \to F \to H \qquad \text{(A2)}$$

A proof of $A \to A$:

1 :	$(A \to (A \to A) \to A) \to (A \to A \to A) \to A \to A$	A2
2 :	$A \to (A \to A) \to A$	A1
3 :	$(A \to A \to A) \to A \to A$	$\to E : 2, 1$
4 :	$A \to A \to A$	A1
5 :	$A \to A$	$\to E : 3, 4$

A simple Hilbert system

Axioms:

$$F \to G \to F \quad \text{(A1)}$$

$$(F \to G \to H) \to (F \to G) \to F \to H \quad \text{(A2)}$$

A proof of $A \to A$:

$$1 : (A \to (A \to A) \to A) \to (A \to A \to A) \to A \to A \qquad \text{A2}$$

$$2 : A \to (A \to A) \to A \qquad \text{A1}$$

$$3 : (A \to A \to A) \to A \to A \qquad {\to}E : 2, 1$$

$$4 : A \to A \to A \qquad \text{A1}$$

$$5 : A \to A \qquad {\to}E : 3, 4$$

$\Rightarrow \vdash_H A \to A$

A simple Hilbert system

Axioms:
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$$3 : (A \to A \to A) \to A \to A \qquad {\to}E : 2, 1$$

$$4 : A \to A \to A \qquad \text{A1}$$

$$5 : A \to A \qquad {\to}E : 3, 4$$

$\Rightarrow \vdash_H A \to A$

Observe: The same proof can be used to derive $F \to F$ for any formula F.

Theorem (Deduction Theorem)

In any Hilbert-system that contains the axioms A1 and A2:

$$F, \Gamma \vdash_H G \quad \textit{iff} \quad \Gamma \vdash_H F \to G$$

Proof "$\Leftarrow$": Assume $\Gamma \vdash_H F \to G$. We prove $F, \Gamma \vdash_H G$.

$$\begin{array}{ll} & \Gamma \vdash_H F \to G \\ \Rightarrow & F, \Gamma \vdash_H F \to G \\ \Rightarrow & F, \Gamma \vdash_H G \qquad \text{by } {\to}E \text{ because } F, \Gamma \vdash_H F \end{array}$$

Theorem (Deduction Theorem)

In any Hilbert-system that contains the axioms A1 and A2:

$$F, \Gamma \vdash_H G \quad \textit{iff} \quad \Gamma \vdash_H F \rightarrow G$$

Proof "$\Rightarrow$": Assume $F, \Gamma \vdash_H G$ with a proof of length n.
We prove $\Gamma \vdash_H F \rightarrow G$ by induction on n.

Base: $n = 1$. Then either $G \in \Gamma \cup \{F\}$ or G is instance of axiom.

- $G = F$. To prove: $\Gamma \vdash_H F \rightarrow F$. Done earlier.
- $G \in \Gamma$ or instance of axiom.

$$\begin{array}{lr} 1: G & \\ 2: G \rightarrow F \rightarrow G & \text{A1} \\ 3: F \rightarrow G & \rightarrow E : 1, 2 \end{array}$$

Theorem (Deduction Theorem)

In any Hilbert-system that contains the axioms A1 and A2:

$$F, \Gamma \vdash_H G \quad \textit{iff} \quad \Gamma \vdash_H F \rightarrow G$$

Proof "$\Rightarrow$": Assume $F, \Gamma \vdash_H G$ with a proof of length n.
We prove $\Gamma \vdash_H F \rightarrow G$ by induction on n.

Step: $n > 1$. Assume last $\rightarrow E$ gives G from $H \rightarrow G$ and H.

IH: $\Gamma \vdash_H F \rightarrow H$ and $\Gamma \vdash_H F \rightarrow H \rightarrow G$.
To prove: $\Gamma \vdash_H F \rightarrow G$.

$$\frac{\qquad\qquad}{F \rightarrow G}$$

Theorem (Deduction Theorem)

In any Hilbert-system that contains the axioms A1 and A2:

$$F, \Gamma \vdash_H G \quad \textit{iff} \quad \Gamma \vdash_H F \to G$$

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To prove: $\Gamma \vdash_H F \rightarrow G$.

$$\dfrac{\dfrac{\qquad\qquad}{(F \rightarrow H) \rightarrow F \rightarrow G} \qquad F \rightarrow H}{F \rightarrow G}$$

Theorem (Deduction Theorem)

In any Hilbert-system that contains the axioms A1 and A2:

$$F, \Gamma \vdash_H G \quad \textit{iff} \quad \Gamma \vdash_H F \to G$$

Proof "⇒": Assume $F, \Gamma \vdash_H G$ with a proof of length n.
We prove $\Gamma \vdash_H F \to G$ by induction on n.

Step: $n > 1$. Assume last $\to E$ gives G from $H \to G$ and H.

IH: $\Gamma \vdash_H F \to H$ and $\Gamma \vdash_H F \to H \to G$.
To prove: $\Gamma \vdash_H F \to G$.

$$\dfrac{\dfrac{(F \to H \to G) \to (F \to H) \to F \to G \qquad F \to H \to G}{(F \to H) \to F \to G} \qquad F \to H}{F \to G}$$

Theorem (Deduction Theorem)

In any Hilbert-system that contains the axioms A1 and A2:

$$F, \Gamma \vdash_H G \quad \textit{iff} \quad \Gamma \vdash_H F \to G$$

Proof "$\Rightarrow$": Assume $F, \Gamma \vdash_H G$ with a proof of length n.
We prove $\Gamma \vdash_H F \to G$ by induction on n.

Step: $n > 1$. Assume last $\to E$ gives G from $H \to G$ and H.

IH: $\Gamma \vdash_H F \to H$ and $\Gamma \vdash_H F \to H \to G$.
To prove: $\Gamma \vdash_H F \to G$.

$$\cfrac{\cfrac{\text{A2: } (F \to H \to G) \to (F \to H) \to F \to G \qquad F \to H \to G}{(F \to H) \to F \to G} \qquad F \to H}{F \to G}$$

Hilbert System

From now on $\vdash_H$ refers to the following set of axioms:

$$F \to G \to F \tag{A1}$$
$$(F \to G \to H) \to (F \to G) \to F \to H \tag{A2}$$
$$F \to G \to F \wedge G \tag{A3}$$
$$F \wedge G \to F \tag{A4}$$
$$F \wedge G \to G \tag{A5}$$
$$F \to F \vee G \tag{A6}$$
$$G \to F \vee G \tag{A7}$$
$$F \vee G \to (F \to H) \to (G \to H) \to H \tag{A8}$$
$$(\neg F \to \bot) \to F \tag{A9}$$

We prove soundness and completeness.

Relating
Hilbert and Natural Deduction

Theorem (Hilbert can simulate ND)

If $\Gamma \vdash_N F$ *then* $\Gamma \vdash_H F$

Proof. Translation in two steps:

$$\vdash_N \quad \overset{(1)}{\leadsto} \quad \vdash_H \; + \rightarrow\!I \quad \overset{(2)}{\leadsto} \quad \vdash_H$$

1. Transform a ND-proof tree into
   a proof tree containing Hilbert axioms, $\rightarrow E$, and $\rightarrow I$
   by replacing all other ND rules by Hilbert proofs with $\rightarrow I$

   Principle: ND rule $\leadsto$ 1 axiom + $\rightarrow I/E$

2. Eliminate $\rightarrow I$ rules using the Deduction Theorem

Theorem (ND can simulate Hilbert)

If $\Gamma \vdash_H F$ *then* $\Gamma \vdash_N F$

Proof by induction on the length of the Hilbert proof of F.

- Every Hilbert axiom is provable in ND (Exercise!).
- $\rightarrow E$ is also available in ND.

Corollary

$\Gamma \vdash_H F$ *iff* $\Gamma \vdash_N F$

Corollary (Soundness and completeness)

$\Gamma \vdash_H F$ *iff* $\Gamma \models F$