Sequent Calculus
Propositional Logic

Sequent Calculus

Invented by Gerhard Gentzen in 1935. Birth of proof theory.

Proof rules

$$\frac{S_1 \quad \dots \quad S_n}{S}$$

where $S_1, \dots S_n$ and S are sequents: expressions of the form

$$\Gamma \Rightarrow \Delta$$

with Γ and Δ finite multisets of formulas.

Multiset = set with possibly repeated elements; using sets possible but less elegant.

Notice: $\Rightarrow$ is just a—suggestive—separator

Intention of the calculus:

$\Gamma \Rightarrow \Delta$ is provable (derivable)

iff

$\bigwedge \Gamma \models \bigvee \Delta \quad (\bigwedge \Gamma \to \bigvee \Delta$ valid)

Sequents: Notation

- We use set notation for multisets, e.g. $\{A, B \to C, A\}$
- Drop $\{\}$: $F_1, \ldots, F_m \Rightarrow G_1, \ldots G_n$
- F, Γ abbreviates $\{F\} \cup \Gamma$ (similarly for Δ)
- Γ_1, Γ_2 abbreviates $\Gamma_1 \cup \Gamma_2$ (similarly for Δ)

Sequent Calculus rules

$$\frac{}{\bot, \Gamma \Rightarrow \Delta}\ \bot L \qquad \frac{}{A, \Gamma \Rightarrow A, \Delta}\ Ax$$

$$\frac{\Gamma \Rightarrow F, \Delta}{\neg F, \Gamma \Rightarrow \Delta}\ \neg L \qquad \frac{F, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \neg F, \Delta}\ \neg R$$

$$\frac{F, G, \Gamma \Rightarrow \Delta}{F \wedge G, \Gamma \Rightarrow \Delta}\ \wedge L \qquad \frac{\Gamma \Rightarrow F, \Delta \quad \Gamma \Rightarrow G, \Delta}{\Gamma \Rightarrow F \wedge G, \Delta}\ \wedge R$$

$$\frac{F, \Gamma \Rightarrow \Delta \quad G, \Gamma \Rightarrow \Delta}{F \vee G, \Gamma \Rightarrow \Delta}\ \vee L \qquad \frac{\Gamma \Rightarrow F, G, \Delta}{\Gamma \Rightarrow F \vee G, \Delta}\ \vee R$$

$$\frac{\Gamma \Rightarrow F, \Delta \quad G, \Gamma \Rightarrow \Delta}{F \to G, \Gamma \Rightarrow \Delta}\ \to L \qquad \frac{F, \Gamma \Rightarrow G, \Delta}{\Gamma \Rightarrow F \to G, \Delta}\ \to R$$

Sequent Calculus rules

Intuition: read backwards as proof search rules

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$$\frac{F, \Gamma \Rightarrow \Delta \quad G, \Gamma \Rightarrow \Delta}{F \vee G, \Gamma \Rightarrow \Delta} \ \vee L$$

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$$\frac{\Gamma \Rightarrow F, \Delta \quad G, \Gamma \Rightarrow \Delta}{F \rightarrow G, \Gamma \Rightarrow \Delta}\ \rightarrow L$$

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$$\frac{F, \Gamma \Rightarrow \Delta \quad G, \Gamma \Rightarrow \Delta}{F \vee G, \Gamma \Rightarrow \Delta}\ \vee L \qquad \frac{\Gamma \Rightarrow F, G, \Delta}{\Gamma \Rightarrow F \vee G, \Delta}\ \vee R$$

$$\frac{\Gamma \Rightarrow F, \Delta \quad G, \Gamma \Rightarrow \Delta}{F \rightarrow G, \Gamma \Rightarrow \Delta}\ \rightarrow L \qquad \frac{F, \Gamma \Rightarrow G, \Delta}{\Gamma \Rightarrow F \rightarrow G, \Delta}\ \rightarrow R$$

Every rule decomposes its principal formula

$$\frac{}{\Rightarrow (P \vee R) \wedge (Q \vee \neg R) \rightarrow P \vee Q}$$

$$\frac{}{\Rightarrow (P \vee R) \wedge (Q \vee \neg R) \rightarrow P \vee Q} \rightarrow R$$

$$\frac{F, \Gamma \Rightarrow G, \Delta}{\Gamma \Rightarrow F \rightarrow G, \Delta} \quad \rightarrow R$$

$$\dfrac{\dfrac{}{(P \vee R) \wedge (Q \vee \neg R) \Rightarrow P \vee Q}}{\Rightarrow (P \vee R) \wedge (Q \vee \neg R) \rightarrow P \vee Q} \rightarrow R$$

$$\dfrac{F, \Gamma \Rightarrow G, \Delta}{\Gamma \Rightarrow F \rightarrow G, \Delta} \quad \rightarrow R$$

$$\dfrac{\dfrac{}{(P \vee R) \wedge (Q \vee \neg R) \Rightarrow P \vee Q}\ \wedge L}{\Rightarrow (P \vee R) \wedge (Q \vee \neg R) \to P \vee Q}\ {\to}R$$

$$\dfrac{F, G, \Gamma \Rightarrow \Delta}{F \wedge G, \Gamma \Rightarrow \Delta} \quad \wedge L$$

$$\cfrac{\cfrac{\overline{P \vee R, Q \vee \neg R \Rightarrow P \vee Q}}{(P \vee R) \wedge (Q \vee \neg R) \Rightarrow P \vee Q} \wedge L}{\Rightarrow (P \vee R) \wedge (Q \vee \neg R) \rightarrow P \vee Q} \rightarrow R$$

$$\frac{F, G, \Gamma \Rightarrow \Delta}{F \wedge G, \Gamma \Rightarrow \Delta} \quad \wedge L$$

$$\cfrac{\cfrac{\overline{P \vee R, Q \vee \neg R \Rightarrow P \vee Q}\ \vee R}{(P \vee R) \wedge (Q \vee \neg R) \Rightarrow P \vee Q}\ \wedge L}{\Rightarrow (P \vee R) \wedge (Q \vee \neg R) \rightarrow P \vee Q}\ \rightarrow R$$

$$\frac{\Gamma \Rightarrow F, G, \Delta}{\Gamma \Rightarrow F \vee G, \Delta} \quad \vee R$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{P \vee R, Q \vee \neg R \Rightarrow P, Q}}{P \vee R, Q \vee \neg R \Rightarrow P \vee Q}\ \vee R}{(P \vee R) \wedge (Q \vee \neg R) \Rightarrow P \vee Q}\ \wedge L}{\Rightarrow (P \vee R) \wedge (Q \vee \neg R) \to P \vee Q}\ \to R$$

$$\dfrac{\Gamma \Rightarrow F, G, \Delta}{\Gamma \Rightarrow F \vee G, \Delta} \qquad \vee R$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad} \ {\vee}L}{P \vee R, Q \vee \neg R \Rightarrow P, Q}}{P \vee R, Q \vee \neg R \Rightarrow P \vee Q} \ {\vee}R}{(P \vee R) \wedge (Q \vee \neg R) \Rightarrow P \vee Q} \ {\wedge}L}{\Rightarrow (P \vee R) \wedge (Q \vee \neg R) \rightarrow P \vee Q} \ {\rightarrow}R$$

$$\dfrac{F, \Gamma \Rightarrow \Delta \quad G, \Gamma \Rightarrow \Delta}{F \vee G, \Gamma \Rightarrow \Delta} \quad {\vee}L$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{P, Q \lor \neg R \Rightarrow P, Q} \qquad \overline{R, Q \lor \neg R \Rightarrow P, Q}}{P \lor R, Q \lor \neg R \Rightarrow P, Q}\ \lor L}{P \lor R, Q \lor \neg R \Rightarrow P \lor Q}\ \lor R}{(P \lor R) \land (Q \lor \neg R) \Rightarrow P \lor Q}\ \land L}{\Rightarrow (P \lor R) \land (Q \lor \neg R) \to P \lor Q}\ \to R$$

$$\frac{F, \Gamma \Rightarrow \Delta \quad G, \Gamma \Rightarrow \Delta}{F \lor G, \Gamma \Rightarrow \Delta} \quad \lor L$$

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{P, Q \lor \neg R \Rightarrow P, Q}\ Ax \qquad \cfrac{}{R, Q \lor \neg R \Rightarrow P, Q}}{P \lor R, Q \lor \neg R \Rightarrow P, Q}\ \lor L}{P \lor R, Q \lor \neg R \Rightarrow P \lor Q}\ \lor R}{(P \lor R) \land (Q \lor \neg R) \Rightarrow P \lor Q}\ \land L}{\Rightarrow (P \lor R) \land (Q \lor \neg R) \to P \lor Q}\ \to R
$$

$$
\cfrac{}{A, \Gamma \Rightarrow A, \Delta} \quad Ax
$$

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{P, Q \lor \neg R \Rightarrow P, Q}\ Ax \qquad \cfrac{}{R, Q \lor \neg R \Rightarrow P, Q}\ \lor L}{P \lor R, Q \lor \neg R \Rightarrow P, Q}\ \lor L}{P \lor R, Q \lor \neg R \Rightarrow P \lor Q}\ \lor R}{(P \lor R) \land (Q \lor \neg R) \Rightarrow P \lor Q}\ \land L}{\Rightarrow (P \lor R) \land (Q \lor \neg R) \to P \lor Q}\ \to R$$

$$\frac{F, \Gamma \Rightarrow \Delta \quad G, \Gamma \Rightarrow \Delta}{F \lor G, \Gamma \Rightarrow \Delta} \quad \lor L$$

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\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P, Q \vee \neg R \Rightarrow P, Q}\ Ax \qquad \dfrac{\dfrac{}{R, Q \Rightarrow P, Q} \qquad \dfrac{}{R, \neg R \Rightarrow P, Q}}{R, Q \vee \neg R \Rightarrow P, Q}\ \vee L}{P \vee R, Q \vee \neg R \Rightarrow P, Q}\ \vee L}{P \vee R, Q \vee \neg R \Rightarrow P \vee Q}\ \vee R}{(P \vee R) \wedge (Q \vee \neg R) \Rightarrow P \vee Q}\ \wedge L}{\Rightarrow (P \vee R) \wedge (Q \vee \neg R) \rightarrow P \vee Q}\ \rightarrow R
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\dfrac{F, \Gamma \Rightarrow \Delta \qquad G, \Gamma \Rightarrow \Delta}{F \vee G, \Gamma \Rightarrow \Delta} \qquad \vee L
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$$

$$
\dfrac{}{A, \Gamma \Rightarrow A, \Delta} \quad Ax
$$

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  \dfrac{
    \dfrac{
      \dfrac{}{P, Q \lor \neg R \Rightarrow P, Q}\,Ax
      \qquad
      \dfrac{
        \dfrac{}{R, Q \Rightarrow P, Q}\,Ax
        \quad
        \dfrac{}{R, \neg R \Rightarrow P, Q}\,\neg L
      }{R, Q \lor \neg R \Rightarrow P, Q}\,\lor L
    }{P \lor R, Q \lor \neg R \Rightarrow P, Q}\,\lor L
  }{
    \dfrac{P \lor R, Q \lor \neg R \Rightarrow P \lor Q}{(P \lor R) \land (Q \lor \neg R) \Rightarrow P \lor Q}\,\land L
  }\,\lor R
}{\Rightarrow (P \lor R) \land (Q \lor \neg R) \to P \lor Q}\,\to R
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\dfrac{\Gamma \Rightarrow F, \Delta}{\neg F, \Gamma \Rightarrow \Delta} \quad \neg L
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\dfrac{
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        \dfrac{}{P, Q \vee \neg R \Rightarrow P, Q}\;Ax
        \qquad
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          \dfrac{}{R, Q \Rightarrow P, Q}\;Ax
          \quad
          \dfrac{\dfrac{}{R \Rightarrow R, P, Q}}{R, \neg R \Rightarrow P, Q}\;\neg L
        }{R, Q \vee \neg R \Rightarrow P, Q}\;\vee L
      }{P \vee R, Q \vee \neg R \Rightarrow P, Q}\;\vee L
    }{P \vee R, Q \vee \neg R \Rightarrow P \vee Q}\;\vee R
  }{(P \vee R) \wedge (Q \vee \neg R) \Rightarrow P \vee Q}\;\wedge L
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      \dfrac{}{P, Q \vee \neg R \Rightarrow P, Q}\ Ax
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        \dfrac{}{R, Q \Rightarrow P, Q}\ Ax
        \qquad
        \dfrac{\dfrac{}{R \Rightarrow R, P, Q}\ Ax}{R, \neg R \Rightarrow P, Q}\ \neg L
      }{R, Q \vee \neg R \Rightarrow P, Q}\ \vee L
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$$

Proof search properties

- For every logical operator ($\neg$ etc)
  there is one left and one right rule
- Every formula in the premise of a rule
  is a subformula of the conclusion of the rule.

  This is called the subformula property.
  $\Rightarrow$ no need to guess anything when applying a rule backward
- Backward rule application terminates
  because one operator is removed in each step.

Instances of rules

Definition

An instance of a rule is the result of replacing Γ and Δ by multisets of concrete formulas and F and G by concrete formulas.

Example

$$\frac{\Rightarrow P \wedge Q, A, B}{\neg(P \wedge Q) \Rightarrow A, B}$$

is an instance of

$$\frac{\Gamma \Rightarrow F, \Delta}{\neg F, \Gamma \Rightarrow \Delta}$$

setting $F := P \wedge Q$, $\Gamma := \emptyset$, $\Delta := \{A, B\}$

Proof trees

Definition (Proof tree)

A proof tree is a tree whose nodes are sequents and where each parent-children fragment

$$\frac{S_1 \quad \dots \quad S_n}{S}$$

is an instance of a proof rule.

($\Rightarrow$ all leaves must be instances of axioms)

A sequent S is provable (or derivable) if there is a proof tree with root S.
We write $\vdash_G S$ to denote that S is derivable.

Proof trees

An alternative inductive definition of proof trees:

Definition (Proof tree)

If

$$\frac{S_1 \quad \dots \quad S_n}{S}$$

is an instance of a proof rule and
there are proof trees $T_1, \dots T_n$ with roots $S_1, \dots, S_n$

then

$$\frac{T_1 \quad \dots \quad T_n}{S}$$

is a proof tree (with root S).

What does $\Gamma \Rightarrow \Delta$ "mean"?

Definition

$$|\Gamma \Rightarrow \Delta| \;=\; \left(\bigwedge \Gamma \to \bigvee \Delta\right)$$

Example: $|\{A, B\} \Rightarrow \{P, Q\}| = (A \wedge B \to P \vee Q)$

Remember: $\bigwedge \emptyset = \top$ and $\bigvee \emptyset = \bot$

In the following slides we prove: $\vdash_G S$ iff $\models |S|$

Soundness

Lemma (Rule Equivalence)

For every rule $\dfrac{S_1 \quad \dots \quad S_n}{S}$

- $|S| \equiv |S_1| \wedge \dots \wedge |S_n|$
- $|S|$ *is a tautology iff all* $|S_i|$ *are tautologies*

Proof: *Exercise.*

Theorem (Soundness of $\vdash_G$)

If $\vdash_G S$ *then* $\models |S|$.

Proof by induction on the height of the proof tree for $\vdash_G S$.
Tree must end in rule instance

$$\frac{S_1 \quad \dots \quad S_n}{S}$$

If $n = 0$ then we vacuously have $\models |S_i|$ for all i.
If $n > 0$ then by IH we also have $\models |S_i|$ for all i.
So $\models |S_i|$ for all i, hence $\models |S|$ by Rule Equivalence.

Proof search = growing a proof tree from the root

To prove completeness we first examine the properties of the proof search procedure:

- Start from an initial sequent S_0
- At each stage we have some potentially *partial* proof tree with unproved leaves
- In each step, pick some unproved leaf S and some rule instance

  $$\frac{S_1 \quad \ldots \quad S_n}{S}$$

  and extend the tree with that rule instance (creating new unproved leaves $S_1, \ldots, S_n$)

Proof search terminates if ...

- there are no more unproved leaves — success
- there is some unproved leaf where no rule applies — failure
  By the rules, that leaf is of the form

$$P_1, \ldots, P_k \Rightarrow Q_1, \ldots, Q_l$$

where all P_i and Q_j are atoms, no $P_i = Q_j$, and no $P_i = \bot$.

Example (failed proof)

$$\frac{}{P \vee Q \Rightarrow P \wedge Q}$$

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Example (failed proof)

$$\frac{}{P \lor Q \Rightarrow P \land Q} \land R$$

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Example (failed proof)

$$\dfrac{\dfrac{}{P \vee Q \Rightarrow P} \qquad \dfrac{}{P \vee Q \Rightarrow Q}}{P \vee Q \Rightarrow P \wedge Q} \wedge R$$

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Example (failed proof)

$$\dfrac{\dfrac{}{P \vee Q \Rightarrow P}\ \vee L \qquad \dfrac{}{P \vee Q \Rightarrow Q}}{P \vee Q \Rightarrow P \wedge Q}\ \wedge R$$

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where all P_i and Q_j are atoms, no $P_i = Q_j$, and no $P_i = \bot$.

Example (failed proof)

$$\dfrac{\dfrac{\dfrac{}{P \Rightarrow P} \qquad Q \Rightarrow P}{P \lor Q \Rightarrow P}\,\lor L \qquad \dfrac{}{P \lor Q \Rightarrow Q}}{P \lor Q \Rightarrow P \land Q}\,\land R$$

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Example (failed proof)

$$\dfrac{\dfrac{\dfrac{}{P \Rightarrow P}\,Ax \qquad Q \Rightarrow P}{P \vee Q \Rightarrow P}\,\vee L \qquad \dfrac{}{P \vee Q \Rightarrow Q}}{P \vee Q \Rightarrow P \wedge Q}\,\wedge R$$

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Example (failed proof)

$$\dfrac{\dfrac{\dfrac{}{P \Rightarrow P}\,Ax \quad Q \Rightarrow P}{P \lor Q \Rightarrow P}\,\lor L \quad \dfrac{P \Rightarrow Q \quad \dfrac{}{Q \Rightarrow Q}\,Ax}{P \lor Q \Rightarrow Q}\,\lor L}{P \lor Q \Rightarrow P \land Q}\,\land R$$

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Example (failed proof)

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Falsifying assignments?

Proof search always terminates

Lemma (Termination)

Proof search terminates from any initial sequent S_0.

Proof

In every step, one logical operator is removed.
$\Rightarrow$ Size of sequent decreases by 1
$\Rightarrow$ Depth of proof tree is bounded by size of S_0
$\Rightarrow$ Construction of proof tree terminates. $\square$

Observe: Breadth only bounded by $2^{\text{size of } S_0}$.

Proof search preserves equivalence

Lemma (Search Equivalence)

At each stage of the search process,
if $S_1, \ldots, S_k$ are the unproved leaves, then $|S_0| \equiv |S_1| \wedge \ldots \wedge |S_k|$

Proof by induction on the number of search steps.
Initially trivially true (base case).
When applying a rule instance

$$\frac{U_1 \quad \ldots \quad U_n}{S_i}$$

we have

$$\begin{aligned} |S_0| \equiv & |S_1| \wedge \ldots \wedge |S_i| \wedge \ldots \wedge |S_k| \\ & \text{(by IH)} \\ \equiv & |S_1| \wedge \cdots \wedge |S_{i-1}| \wedge |U_1| \wedge \cdots \wedge |U_n| \wedge |S_{i+1}| \wedge \ldots \wedge |S_k| \\ & \text{(by Lemma Rule Equivalence)} \end{aligned}$$

Completeness

Lemma

If proof search fails, $|S_0|$ is not a tautology.

Proof If proof search fails, there is some unproved leaf

$$S = \qquad P_1, \ldots, P_k \Rightarrow Q_1, \ldots, Q_l$$

where all P_i, Q_j atoms, no $P_i = Q_j$ and no $P_i = \bot$.
Any assignment $\mathcal{A}$ with $\mathcal{A}(P_i) = 1$ (for all i)
and $\mathcal{A}(Q_j) = 0$ (for all j) satisfies $\mathcal{A}(|S|) = 0$.
Thus $\mathcal{A}(|S_0|) = 0$ by Lemma Search Equivalence. □

Because of soundness of $\vdash_G$:

Corollary

Starting with some fixed S_0, proof search cannot both fail (for some choices) and succeed (for other choices).

$\Rightarrow$ no need for backtracking upon failure!

Completeness

Theorem (Completeness)

If $\models |S|$ *then* $\vdash_G S$*.*

Proof by contraposition: if not $\vdash_G S$ then proof search must fail. Therefore $\not\models |S|$.

Corollary

Proof search is a decision procedure: it always terminates and it succeeds iff $\models S$*.*

Multisets versus sets

Termination only because of multisets.
With sets, the principal formula may get duplicated:

$$\frac{\Gamma \Rightarrow F, \Delta}{\neg F, \Gamma \Rightarrow \Delta} \quad \neg L \qquad \overset{\Gamma := \{\neg F\}}{\rightsquigarrow} \qquad \frac{\neg F \Rightarrow F, \Delta}{\neg F \Rightarrow \Delta}$$

An alternative formulation of the set version:

$$\frac{\Gamma \setminus \{\neg F\} \Rightarrow F, \Delta}{\neg F, \Gamma \Rightarrow \Delta}$$

Gentzen used sequences (hence "sequent calculus")

Admissible Rules and Cut Elimination

Admissible rules

Definition

A rule

$$\frac{S_1 \quad \dots \quad S_n}{S}$$

is admissible if $\vdash_G S_1, \dots, \vdash_G S_n$ together imply $\vdash_G S$.

$\Rightarrow$ Admissible rules can be used in proofs like normal rules

Admissibility of

$$\frac{S_1 \quad \dots \quad S_n}{S}$$

can be shown semantically (using $\vdash_G$ iff $\models$)
by proving that $\models |S_1|, \dots, \models |S_n|$ together imply $\models |S|$.

Proof theory is interested in syntactic proofs that show how to eliminate admissible rules.

Cut elimination rule

Theorem (Gentzen's *Hauptsatz*)

The cut elimination rule

$$\frac{\Gamma \Rightarrow F, \Delta \quad \Gamma, F \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \quad \textit{cut}$$

is admissible.

Proof Omitted.

Proofs with cut elimination can be much shorter than proofs without!
But: applying the rule needs creativity! (find the right F)
Intuitively: Proof of Gentzen's theorem shows how to replace creativity by calculation.

Many applications.