Basic Proof Theory
Propositional Logic

(See the book by Troelstra and Schwichtenberg)

Proof rules and proof systems

Proof systems are defined by (proof or inference) rules of the form

$$\frac{T_1 \quad \dots \quad T_n}{T} \text{ rule-name}$$

where $T_1, \dots T_n$ (premises) and T (conclusion) are syntactic objects (eg formulas).

Intuitive reading: If $T_1, \dots, T_n$ are provable, then T is provable.

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Degenerate case: If $n = 0$ the rule is called an axiom and the horizontal line is sometimes omitted.

If some U is provable, we write $\vdash U$.

Proof trees

Proofs (also: derivations) are drawn as trees of nested proof rules. Example:

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Every fragment

$$\frac{T_1 \quad \dots \quad T_n}{T}$$

of a proof tree must be (an instance of) a proof rule.
All proofs must start with axioms.

The depth of a proof tree is the number of rules on the longest branch of the tree. Thus ≥ 1

Abbreviations

Until further notice:

$\bot$, $\neg$, $\wedge$, $\vee$, $\rightarrow$ are primitives.

$\top$ abbreviates $\neg\bot$

A possible simplification:

$\neg F$ abbreviates $F \rightarrow \bot$

We now consider three important proof systems:

- Sequent Calculus
- Natural Deduction
- Hilbert Systems