Propositional Logic
Resolution

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Question: Is F unsatisfiable?

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If the empty clause is found, the initial F is unsatisfiable
Completeness:
If the initial F is unsatisfiable, the empty clause can be found.

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Correctness/Completeness of syntactic procedure (resolution) w.r.t. semantic property (unsatisfiability)

Resolvent

Definition

Let L be a literal. Then $\overline{L}$ is defined as follows:

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Definition

Let C_1, C_2 be clauses and let L be a literal such that $L \in C_1$ and $\overline{L} \in C_2$. Then the clause

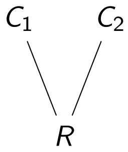
$$(C_1 - \{L\}) \cup (C_2 - \{\overline{L}\})$$

is a resolvent of C_1 and C_2.
The process of deriving the resolvent is called a resolution step.

Graphical representation of resolvent:

$$
\begin{tikzcd}[column sep=small]
C_1 \arrow[dr, dash] & & C_2 \arrow[dl, dash] \\
& R &
\end{tikzcd}
$$

Graphical representation of resolvent:



If $C_1 = \{L\}$ and $C_2 = \{\overline{L}\}$ then the empty clause is a resolvent of C_1 and C_2. The special symbol $\square$ denotes the empty clause.

Recall: $\square$ represents $\perp$.

Resolution proof

Definition

A resolution proof of a clause C from a set of clauses F is a sequence of clauses $C_0, \dots, C_n$ such that

- $C_i \in F$ or C_i is a resolvent of two clauses C_a and C_b, $a, b < i$,
- $C_n = C$

Then we can write $F \vdash_{Res} C$.

Note: F can be finite or infinite!

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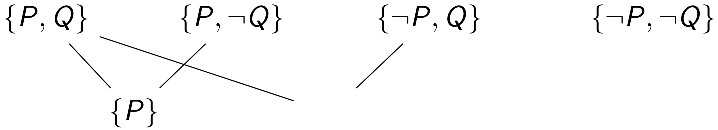

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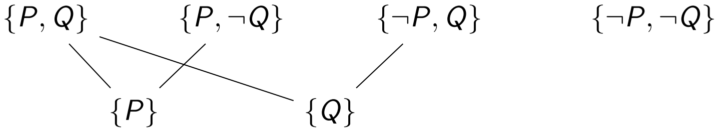

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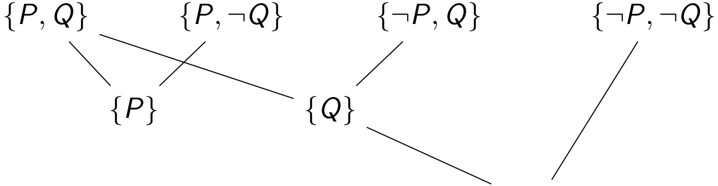

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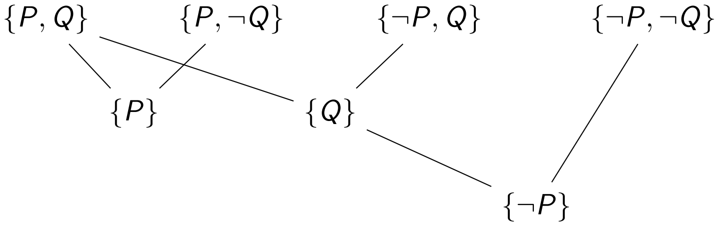

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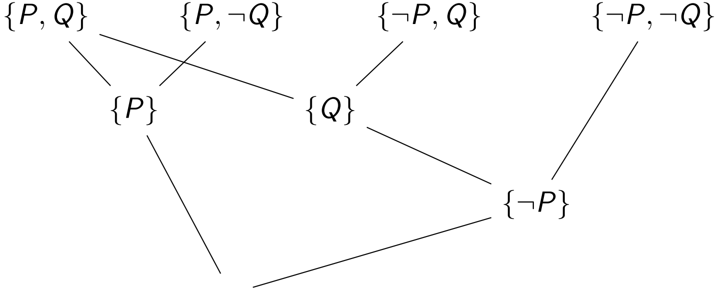

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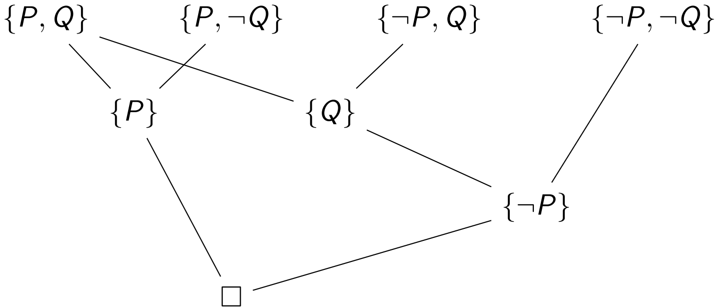

A linear resolution proof

0:	$\{P, Q\}$	
1:	$\{P, \neg Q\}$	
2:	$\{\neg P, Q\}$	
3:	$\{\neg P, \neg Q\}$	
4:	$\{P\}$	(0, 1)
5:	$\{Q\}$	(0, 2)
6:	$\{\neg P\}$	(3, 5)
7:	$\square$	(4, 6)

Correctness of resolution

Lemma (Resolution Lemma)

Let R be a resolvent of two clauses C_1 and C_2.

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Proof By definition $R = (C_1 - \{L\}) \cup (C_2 - \{\overline{L}\})$ (for some L).
Assume $\mathcal{A} \models C_1$ and $\mathcal{A} \models C_2$. We show $\mathcal{A} \models R$.
There are two cases:

- $\mathcal{A} \models L$. Then $\mathcal{A} \models C_2 - \{\overline{L}\}$ (because $\mathcal{A} \models C_2$),

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- $C_i \in F$.
  Then $F \models C_i$ by definition.
- C_i is a resolvent of C_a and C_b for $a, b < i$.
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Corollary

Let F be a set of clauses. If $F \vdash_{Res} \square$ then F is unsatisfiable.

Completeness of resolution

Theorem

Let F be a finite set of clauses. If F is unsatisfiable then $F \vdash_{Res} \square$.

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Corollary

A set of clauses F is unsatisfiable iff $F \vdash_{Res} \Box$.

Completeness proof

Corollary

(of the Boole-Shannon expansion) F is unsatisfiable iff $F[\bot/A]$ and $F[\top/A]$ are unsatisfiable.

Idea for completeness proof:
If A is an atom of F, then both $F[\bot/A]$ and $F[\top/A]$ have fewer atoms than F.

Use Boole-Shannon to prove completeness by induction on the number of atoms of the unsatisfiable formula F:

- construct inductively resolution proofs for $F[\bot/A]$ and $F[\top/A]$, and
- "combine" them into a resolution proof for F.

Inductive construction of resolution proofs

$$F = \{\ \{\neg q, s\}\ ,\ \{\neg p, q, s\}\ ,\ \{p\}\ ,\ \{r, \neg s\}\ ,\ \{\neg p, \neg r, \neg s\}\ \}$$

- Compute inductively proofs for $F[\top/s]$ and $F[\bot/s]$.

$$\begin{aligned} F[\top/s] &\equiv \{\ \{p\}\ ,\ \{r\}\ ,\ \{\neg p, \neg r\}\ \} \\ F[\bot/s] &\equiv \{\ \{\neg q\}\ ,\ \{\neg p, q\}\ ,\ \{p\}\ \} \end{aligned}$$

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$$\begin{tikzcd} \{p\} \arrow[drr] & \{r\} \arrow[dd] & & \{\neg p, \neg r\} \arrow[dl] \\ & & \{\neg r\} \arrow[dl] & \\ & \square & & \end{tikzcd} \qquad \begin{tikzcd} \{\neg q\} \arrow[ddrr] & & \{q, \neg p\} \arrow[dr] & & \{p\} \arrow[dl] \\ & & & \{q\} \arrow[dl] & \\ & & \square & & \end{tikzcd}$$

Inductive construction of resolution proofs

- Reintroduce s and $\neg s$.

$$F = \{\ \{\neg q, s\}\ ,\ \{\neg p, q, s\}\ ,\ \{p\}\ ,\ \{r, \neg s\}\ ,\ \{\neg p, \neg r, \neg s\}\ \}$$

$$\begin{tikzcd}
\{\neg q\} \arrow[ddr] & \{\neg p, q\} \arrow[dr] & & \{p\} \arrow[dl] \\
& & \{q\} \arrow[dl] & \\
& \square & &
\end{tikzcd}
\qquad
\begin{tikzcd}
\{p\} \arrow[drr] & \{r\} \arrow[dd] & & \{\neg p, \neg r\} \arrow[dl] \\
& & \{\neg r\} \arrow[dl] & \\
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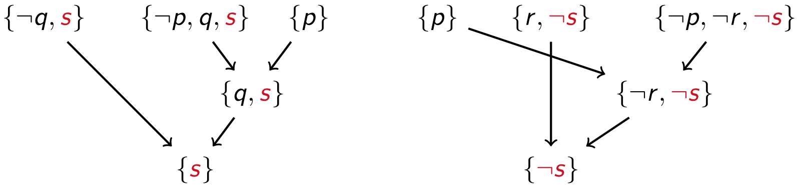
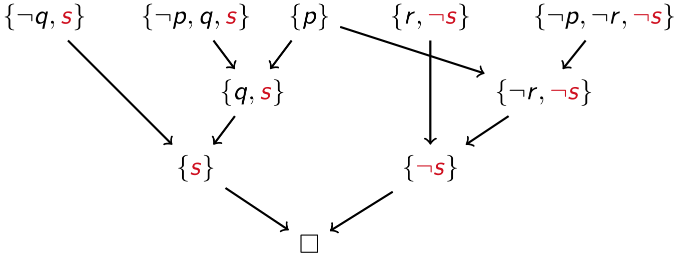
\end{tikzcd}$$

$$\begin{tikzcd}
\{\neg q, s\} \arrow[ddr] & \{\neg p, q, s\} \arrow[dr] & & \{p\} \arrow[dl] \\
& & \{q, s\} \arrow[dl] & \\
& \{s\} & &
\end{tikzcd}
\qquad
\begin{tikzcd}
\{p\} \arrow[drr] & \{r, \neg s\} \arrow[dd] & & \{\neg p, \neg r, \neg s\} \arrow[dl] \\
& & \{\neg r, \neg s\} \arrow[dl] & \\
& \{\neg s\} & &
\end{tikzcd}$$

Inductive construction of resolution proofs

- Combine the graphs for $\{s\}$ and $\{\neg s\}$.





Completeness proof

Theorem

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Theorem

Let F be a finite set of clauses. If F is unsatisfiable then $F \vdash_{Res} \Box$.

Proof By induction on the number n of distinct atoms in F.

Basis: If $n = 0$ then $F = \{\}$ (but F is unsat.)

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Step:
IH: For every unsat. set of clauses F with n dist. atoms, $F \vdash_{Res} \Box$.
Let F contain $n + 1$ distinct atoms. Pick some atom A in F.

$F[\top/A] \equiv$ take F, remove all clauses with A, remove all $\neg A$.
$F[\bot/A] \equiv$ take F, remove all clauses with $\neg A$, remove all A.

Completeness proof

By IH: there are res. proofs $C_0, \dots, C_m = \Box$ from $F[\bot/A]$ and $D_0, \dots, D_n = \Box$ from $F[\top/A]$.

Now transform $C_0, \dots, C_m$ into a proof $C'_0, \dots, C'_m$ from F by adding A back into the clauses it was removed from. Then:

- either $C'_m = \{A\}$
- or $C'_m = \Box$ (and we are done).

Similarly we transform $D_0, \dots, D_n$ into a proof $D'_0, \dots, D'_n$ from F by adding $\neg A$ back in. Then:

- either $D'_n = \{\neg A\}$
- or $D'_n = \Box$ (and we are done).

If $C'_m = \{A\}$ and $D'_n = \{\neg A\}$ then $F \vdash_{Res} A$ and $F \vdash_{Res} \neg A$ and thus $F \vdash_{Res} \Box$.

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Exercise

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Prove $F \cup \{\neg C\} \vdash_{Res} \Box$

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do $F := F \cup \{R\}$

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The algorithm terminates.

Proof There are only finitely many clauses over a finite set of atoms.

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The initial F is unsatisfiable iff $\Box$ is in the final F

Proof F_{init} is unsat. iff $F_{init} \vdash_{Res} \Box$ iff $\Box \in F_{final}$ because the algorithm enumerates all R such that $F_{init} \vdash_{Res} R$.

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The algorithm is a decision procedure for unsat. of CNF formulas.