# Propositional Logic
# Compactness

# Compactness Theorem

Theorem

*A set $S$ of formulas is satisfiable*
*iff every finite subset of $S$ is satisfiable.*

Equivalent formulation:

A set $S$ of formulas is unsatisfiable
iff some finite subset of $S$ is unsatisfiable.

# An application: Graph Coloring

**Definition**

A 4-coloring of a graph $(V, E)$ is a map $c : V \to \{1, 2, 3, 4\}$ such that $(x, y) \in E$ implies $c(x) \neq c(y)$.

**Theorem (4CT)**

*A finite planar graph has a 4-coloring.*

**Theorem**

*A planar graph $G = (V, E)$ with countably many vertices $V = \{v_1, v_2, \ldots\}$ has a 4-coloring.*

**Proof** $G \rightsquigarrow$ set of formulas $S$ s.t. $S$ is sat. iff $G$ is 4-col.

$G$ is planar

$\Rightarrow$ every finite subgraph of $G$ is planar and 4-col. (by 4CT)

$\Rightarrow$ every finite subset of $S$ is sat.

$\Rightarrow$ $S$ is sat. (by Compactness)

$\Rightarrow$ $G$ is 4-col.

# Proof details

$G \rightsquigarrow S$:

For simplicity:
atoms are of the form $A_i^c$ where $c \in \{1, \ldots, 4\}$ and $i \in \mathbb{N}$

$$
\begin{aligned}
S := \quad & \{A_i^1 \vee A_i^2 \vee A_i^3 \vee A_i^4 \mid i \in \mathbb{N}\} \cup \\
& \{A_i^c \to \neg A_i^d \mid i \in \mathbb{N},\ c, d \in \{1, \ldots, 4\},\ c \neq d\} \cup \\
& \{\neg(A_i^c \wedge A_j^c) \mid (v_i, v_j) \in E,\ c \in \{1, \ldots, 4\}\}
\end{aligned}
$$

Subgraph corresponding to some $T \subseteq S$:
$V_T := \{v_i \mid A_i^c \text{ occurs in } T \text{ (for some } c)\}$
$E_T := \{(v_i, v_j) \mid \neg(A_i^c \wedge A_j^c) \in T \text{ (for some } c)\}$

# Proof of Compactness

Theorem
*A set $S$ of formulas is satisfiable*
*iff every finite subset of $S$ is satisfiable.*

**Proof**

$\Rightarrow$: If $S$ is satisfiable then every finite subset of $S$ is satisfiable.

Trivial.

$\Leftarrow$ : If every finite subset of $S$ is satisfiable then $S$ is satisfiable.

We prove that $S$ has a model.

# Proof of Compactness

## Definition

Let $b_1 \cdots b_n \in \{0,1\}^*$ with $n \geq 0$ and let $T$ be a set of formulas.
An assignment $\mathcal{A}$ is a $b_1 \cdots b_n$–model of $T$ if $\mathcal{A}(A_i) = b_i$ for every $i = 1, \ldots, n$ and $\mathcal{A} \models T$.

In particular: every model is a $\varepsilon$-model.

Assume every finite $T \subseteq S$ is satisfiable. We prove:

1. There is an infinite sequence $b_1 b_2 \cdots \in \{0,1\}^\omega$ such that for every $n \geq 1$ all finite $T \subseteq S$ have a $b_1 \cdots b_n$–model.
2. The assignment $\mathcal{B}$ given by $\mathcal{B}(A_i) := b_i$ for all $i \geq 1$ is a model of $S$.

# Proof of Compactness: Part (1)

To prove: There is an infinite sequence $b_1 b_2 \cdots \in \{0,1\}^\omega$ such that for every $n \geq 1$ all finite $T \subseteq S$ have a $b_1 \cdots b_n$–model.

It suffices to show:

(a) Every finite $T \subseteq S$ has an $\varepsilon$–model.

(b) For every sequence $\sigma \in \{0,1\}^*$: if every finite $T \subseteq S$ has a $\sigma$–model then there exists $b \in \{0,1\}$ such that every finite $T \subseteq S$ has a $\sigma b$–model.

**Proof of (a)**: By assumption every $T \subseteq S$ has an $\varepsilon$–model.

**Proof of (b)**: Next slide.

# Proof of Compactness: Part (1)

**Proof of (b)**: By contradiction.

Assume that for some $\sigma \in \{0,1\}$: every finite $T \subseteq S$ has a $\sigma$–model, but

(0) some finite $T_0 \subseteq S$ has no $\sigma 0$–model; and

(1) some finite $T_1 \subseteq S$ has no $\sigma 1$–model.

Consider the finite set $T_0 \cup T_1$.
By assumption, $T_0 \cup T_1$ has some $\sigma$–model $\mathcal{A}$. Let $n := |\sigma|$.

Two possible cases:

- $\mathcal{A}(A_{n+1}) = 0$. Then $\mathcal{A}$ is a $\sigma 0$–model of $T_0$, contradicting (0).
- $\mathcal{A}(A_{n+1}) = 1$. Then $\mathcal{A}$ is a $\sigma 1$–model of $T_1$, contradicting (1).

# Proof of Compactness: Part (2)

To prove: The assignment $\mathcal{B}$ given by $\mathcal{B}(A_i) := b_i$ for all $i \geq 1$, where $b_1 b_2 \cdots$ is the infinite sequence of (1), is a model of $S$.

We show $\mathcal{B} \models F$ for all $F \in S$.

Let $m$ be the maximal index of all atoms in $F$.
By (1), $\{F\}$ has a $b_1 \cdots b_m$–model $\mathcal{A}$.

Hence $\mathcal{B} \models F$, because $\mathcal{A}$ and $\mathcal{B}$ agree on all atoms in $F$.

# Corollary

**Corollary**

*If $S \models F$ then there is a finite subset $M \subseteq S$ such that $M \models F$.*