Propositional Logic
Horn Formulas

Efficient satisfiability checks

In this and the next slide sets:

- A very efficient satisfiability check for a special class of formulas in CNF: Horn formulas,
- Efficient satisfiability checks for arbitrary formulas in CNF: DPLL and resolution (later).

Horn formulas

Definition

A formula F in CNF is a Horn formula if every disjunction in F contains at most one positive literal.

Every disjunct of a Horn formula can equivalently be viewed as an implication $K \to B$ where

- K is a conjunction of atoms or $\top$, and
- B is an atom or $\bot$.

$$
\begin{array}{rcl}
A & \equiv & (\top \to A) \\
(\neg A \vee \neg B \vee C) & \equiv & (A \wedge B \to C) \\
(\neg A \vee B) & \equiv & (A \to B) \\
\neg A & \equiv & (A \to \bot) \\
(\neg A \vee \neg B) & \equiv & (A \wedge B \to \bot)
\end{array}
$$

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$$
\begin{array}{rcll}
A & \equiv & (\top \to A) & \text{fact} \\
(\neg A \vee \neg B \vee C) & \equiv & (A \wedge B \to C) & \text{rule} \\
(\neg A \vee B) & \equiv & (A \to B) & \text{rule} \\
\neg A & \equiv & (A \to \bot) & \text{goal} \\
(\neg A \vee \neg B) & \equiv & (A \wedge B \to \bot) & \text{goal}
\end{array}
$$

Satisfiability check for Horn formulas

Horn: $(\neg A \vee \neg B \vee C) \equiv (A \wedge B \rightarrow C)$

Non-Horn: $(\neg A \vee \neg B \vee C \vee D) \equiv (A \wedge B \rightarrow C \vee D)$

Satisfiability check for Horn formulas

Input: a Horn formula F.
Output: Model $\mathcal{M}$ of F or "unsatisfiable"

for all atoms A_i in F **do** $\mathcal{M}(A_i) := 0$;
while F has a conjunct $K \to B$
such that $\mathcal{M}(K) = 1$ and $\mathcal{M}(B) = 0$
do
if $B = \bot$ **then return** "unsatisfiable"
else $\mathcal{M}(B) := 1$

return $\mathcal{M}$

Maximal number of iterations of the while loop:
number of implications in F

Each iteration requires at most $O(|F|)$ steps.

Overall complexity: $O(|F|^2)$

[Algorithm can be improved to $O(|F|)$. See Schöning.]

Correctness of the model building algorithm

Theorem

The algorithm returns a model iff F is satisfiable.

Proof. Invariant: if $\mathcal{M}(A) = 1$, then $\mathcal{A}(A) = 1$ for every atom A and model $\mathcal{A}$ of F.

(a) If "unsatisfiable" then unsatisfiable.
Assume F has model $\mathcal{A}$ but algorithm answers "unsatisfiable".
Let $(A_{i_1} \wedge \ldots \wedge A_{i_k} \rightarrow \bot)$ be the subformula causing "unsatisfiable".
Since $\mathcal{M}(A_{i_1}) = \cdots = \mathcal{M}(A_{i_k}) = 1$, $\mathcal{A}(A_{i_1}) = \ldots = \mathcal{A}(A_{i_k}) = 1$.
Then $\mathcal{A}(A_{i_1} \wedge \ldots \wedge A_{i_k} \rightarrow \bot) = 0$ and so $\mathcal{A}(F) = 0$, contradiction.

(b) If "$\mathcal{M}$" then $\mathcal{M} \models F$.
After termination with "$\mathcal{M}$", every conjunct $K \rightarrow B$ of F satisfies $\mathcal{M}(K) = 0$ or $\mathcal{M}(B) = 1$.
Therefore $\mathcal{M}(K \rightarrow B) = 1$ and thus $\mathcal{M} \models F$.

Correctness of the model building algorithm

Corollary

A satisfiable Horn formula has a unique model with a smallest number of true atoms.