# Propositional Logic

# Definitional CNF

## (Tseytin's transformation)

# Definitional CNF

1. The definitional CNF of a formula is obtained in 2 steps:

   Repeatedly replace a subformula $G$ of the form $\neg A$, $A \wedge B$ or $A \vee B$ ($A, B$ atoms!) by a new atom $A'$ and conjoin $A' \leftrightarrow G$.

   (This replacement is not applied to the "definitions" $A' \leftrightarrow G$ but only to the (remains of the) original formula.)
2. Translate all the subformulas $A' \leftrightarrow G$ into CNF.

## Example

$$\neg(\boxed{A_1 \vee A_2}) \wedge A_3$$
$$\rightsquigarrow \quad \boxed{\neg A_4} \wedge A_3 \wedge \ (A_4 \leftrightarrow (A_1 \vee A_2))$$
$$\rightsquigarrow \quad \boxed{A_5 \wedge A_3} \wedge \ (A_4 \leftrightarrow (A_1 \vee A_2)) \wedge (A_5 \leftrightarrow \neg A_4)$$
$$\rightsquigarrow \quad A_6 \wedge \ (A_4 \leftrightarrow (A_1 \vee A_2)) \wedge (A_5 \leftrightarrow \neg A_4) \wedge (A_6 \leftrightarrow (A_5 \wedge A_3))$$
$$\rightsquigarrow \quad A_6 \wedge CNF(A_4 \leftrightarrow (A_1 \vee A_2)) \wedge CNF(A_5 \leftrightarrow \neg A_4) \wedge CNF(A_6 \leftrightarrow (A_5 \wedge A_3))$$

# Definitional CNF: Complexity

Let the initial formula have size $n$.

1. Each replacement step increases the size of the formula by a constant.
   There are at most as many replacement steps as subformulas, linearly many.
2. The conversion of each $A \leftrightarrow G$ into CNF increases the size by a constant.
   There are only linearly many such subformulas.

Thus: the definitional CNF has size $O(n)$, and
can be constructed in $O(n)$ time.

# Definitional CNF: Correctness — Notation

**Definition**

The notation $F[G/A]$ denotes the result of replacing all occurrences of the atom $A$ in $F$ by $G$.
We pronounce it as "$F$ with $G$ for $A$".

**Example**

$(A \wedge B)[(A \rightarrow B)/B] = (A \wedge (A \rightarrow B))$

**Definition**

The notation $\mathcal{A}[v/A]$ denotes a modified version of $\mathcal{A}$ that maps $A$ to $v$ and behaves like $\mathcal{A}$ otherwise:

$$(\mathcal{A}[v/A])(A_i) = \begin{cases} v & \text{if } A_i = A \\ \mathcal{A}(A_i) & \text{otherwise} \end{cases}$$

# Definitional CNF: Correctness — Substitution Lemma

Lemma

$\mathcal{A}(F[G/A]) \;=\; \mathcal{A}'(F)$ *where* $\mathcal{A}' = \mathcal{A}[\mathcal{A}(G)/A]$

**Proof** by structural induction on $F$.

- $F$ is an atom:
  If $F = A$: $\mathcal{A}(F[G/A]) = \mathcal{A}(G) = \mathcal{A}'(F)$
  If $F \neq A$: $\mathcal{A}(F[G/A]) = \mathcal{A}(F) = \mathcal{A}'(F)$
- $F = F_1 \wedge F_2$:

$$\begin{aligned}
\mathcal{A}((F_1 \wedge F_2)[G/A]) &= \mathcal{A}(F_1[G/A] \wedge F_2[G/A]) \\
&= \min(\mathcal{A}(F_1[G/A]), \mathcal{A}(F_2[G/A])) \\
&\overset{IH}{=} \min(\mathcal{A}'(F_1), \mathcal{A}'(F_2)) \\
&= \mathcal{A}'(F_1 \wedge F_2)
\end{aligned}$$

# Definitional CNF: Correctness

Each replacement step produces an equisatisfiable formula:

## Lemma

*Let $A$ be an atom that does not occur in $G$.*
*Then $F[G/A]$ is equisatisfiable with $F \wedge (A \leftrightarrow G)$.*

**Proof** Assume $\mathcal{A} \models F[G/A]$ for some assignment $\mathcal{A}$.
Let $\mathcal{A}' := \mathcal{A}[\mathcal{A}(G)/A]$. We prove $\mathcal{A}' \models F \wedge (A \leftrightarrow G)$.
$\mathcal{A}' \models F$: Substitution Lemma.
$\mathcal{A}' \models (A \leftrightarrow G)$: Because $\mathcal{A}'(A) = \mathcal{A}(G) = \mathcal{A}'(G)$
(by definition of $\mathcal{A}'$ and because $A$ does not occur in $G$).

Assume $\mathcal{A} \models F \wedge (A \leftrightarrow G)$ for some assignment $\mathcal{A}$.
We prove $\mathcal{A} \models F[G/A]$, that is, $\mathcal{A}(F[G/A]) = 1$.
We show $\mathcal{A}(F[G/A]) = \mathcal{A}'(F) = \mathcal{A}(F) = 1$ for $\mathcal{A}' := \mathcal{A}[\mathcal{A}(G)/A]$.
$\mathcal{A}(F[G/A]) = \mathcal{A}'(F)$: Substitution Lemma.
$\mathcal{A}'(F) = \mathcal{A}(F)$: From $\mathcal{A} \models (A \leftrightarrow G)$ follows $\mathcal{A}(A) = \mathcal{A}(G)$, and so $\mathcal{A}' = \mathcal{A}$.
$\mathcal{A}(F) = 1$: Because $\mathcal{A} \models F$.

# Definitional CNF: Correctness

Does $F \wedge (A \leftrightarrow G) \models F[G/A]$ hold?

Does $F[G/A] \models F \wedge (A \leftrightarrow G)$ hold?

# Summary

**Theorem**

*For every formula $F$ of size $n$*
*there is an equisatisfiable CNF formula $G$ of size $O(n)$.*

**Proof.**

Repeated application of the Lemma. □

Similarly it can be shown:

**Theorem**

*For every formula $F$ of size $n$*
*there is an equivalid DNF formula $G$ of size $O(n)$.*

# Validity of CNF

Validity of formulas in CNF can be checked in linear time.

*A formula in CNF is valid iff all its disjunctions are valid.*
*A disjunction is valid iff it contains both an atomic A and* $\neg A$ *as literals.*

Example

Valid: $(A \vee \neg A \vee B) \wedge (C \vee \neg C)$

Not valid: $(A \vee \neg A) \wedge (\neg A \vee C)$

# Satisfiability of DNF

Satisfiability of formulas in DNF can be checked in linear time.

*A formula in DNF is satisfiable iff at least one of its conjunctions is satisfiable. A conjunction is satisfiable iff it does not contain both an atomic $A$ and $\neg A$ as literals.*

**Example**

Satisfiable: $(\neg B \wedge A \wedge B) \vee (\neg A \wedge C)$

Unsatisfiable: $(A \wedge \neg A \wedge B) \vee (C \wedge \neg C)$

# Satisfiability/validity of DNF and CNF

**Theorem**
*Satisfiability of formulas in CNF is NP-complete.*

**Theorem**
*Validity of formulas in DNF is co-NP-complete.*

Standard decision procedure for validity of $F$:

1. Transform $\neg F$ into an equisat. formula $G$ in def. CNF
2. Apply efficient CNF-based SAT solver to $G$