Propositional Logic
Normal Forms

Abbreviations

Until further notice:

$F_1 \to F_2$	abbreviates	$\neg F_1 \lor F_2$
$F_1 \leftrightarrow F_2$	abbreviates	$(F_1 \land F_2) \lor (\neg F_1 \land \neg F_2)$
$\top$	abbreviates	$A_1 \lor \neg A_1$
$\bot$	abbreviates	$A_1 \land \neg A_1$

Literals

Definition

A literal is an atom or the negation of an atom.
In the former case the literal is positive,
in the latter case it is negative.

Negation Normal Form (NNF)

Definition

A formula is in negation formal form (NNF)
if negation ($\neg$) occurs only directly in front of atoms.

Example

In NNF: $\neg A \wedge \neg B$

Not in NNF: $\neg(A \vee B)$

Transformation into NNF

Any formula can be transformed into an equivalent formula in NNF by pushing $\neg$ inwards. Apply the following equivalences from left to right as long as possible:

$$\begin{aligned} \neg\neg F &\equiv F \\ \neg(F \wedge G) &\equiv (\neg F \vee \neg G) \\ \neg(F \vee G) &\equiv (\neg F \wedge \neg G) \end{aligned}$$

Example

$(\neg(A \wedge \neg B) \wedge C) \;\equiv\; ((\neg A \vee \neg\neg B) \wedge C) \;\equiv\; ((\neg A \vee B) \wedge C)$

("$F \equiv G \equiv H$" is an abbreviation for "$F \equiv G$ and $G \equiv H$")

Does this process always terminate? Is the result unique?

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CNF and DNF

Definition

A formula F is in conjunctive normal form (CNF) if it is a conjunction of disjunctions of literals:

$$F = (\bigwedge_{i=1}^{n} (\bigvee_{j=1}^{m_i} L_{i,j})),$$

where $L_{i,j} \in \{A_1, A_2, \cdots\} \cup \{\neg A_1, \neg A_2, \cdots\}$

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Definition

A formula F is in disjunctive normal form (DNF) if it is a disjunction of conjunctions of literals:

$$F = (\bigvee_{i=1}^{n} (\bigwedge_{j=1}^{m_i} L_{i,j})),$$

where $L_{i,j} \in \{A_1, A_2, \cdots\} \cup \{\neg A_1, \neg A_2, \cdots\}$

Transformation into CNF and DNF

Any formula can be transformed into an equivalent formula in CNF or DNF in two steps:

1. Transform the initial formula into its NNF
2. Transform the NNF into CNF or DNF:
   - Transformation into CNF. Apply the following equivalences from left to right as long as possible:

$$
\begin{aligned}
(F \vee (G \wedge H)) &\equiv ((F \vee G) \wedge (F \vee H)) \\
((F \wedge G) \vee H) &\equiv ((F \vee H) \wedge (G \vee H))
\end{aligned}
$$

   - Transformation into DNF. Apply the following equivalences from left to right as long as possible:

$$
\begin{aligned}
(F \wedge (G \vee H)) &\equiv ((F \wedge G) \vee (F \wedge H)) \\
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$$
\begin{aligned}
(F \lor (G \land H)) &\equiv ((F \lor G) \land (F \lor H)) \\
((F \land G) \lor H) &\equiv ((F \lor H) \land (G \lor H))
\end{aligned}
$$

   - Transformation into DNF. Apply the following equivalences from left to right as long as possible:

$$
\begin{aligned}
(F \land (G \lor H)) &\equiv ((F \land G) \lor (F \land H)) \\
((F \lor G) \land H) &\equiv ((F \land H) \lor (G \land H))
\end{aligned}
$$

Termination

Why does the transformation into NNF and CNF terminate?

Challenge Question: Find a weight function $w :: \textit{formula} \to \mathbb{N}$ such that $w(l.h.s.) > w(r.h.s.)$ for the equivalences

$$
\begin{array}{rcl}
\neg\neg F & \equiv & F \\
\neg(F \wedge G) & \equiv & (\neg F \vee \neg G) \\
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(F \vee (G \wedge H)) & \equiv & ((F \vee G) \wedge (F \vee H)) \\
((F \wedge G) \vee H) & \equiv & ((F \vee H) \wedge (G \vee H))
\end{array}
$$

Define w recursively:

$w(A_i) = \ldots$

$w(\neg F) = \ldots w(F) \ldots$

$w(F \wedge G) = \ldots w(F) \ldots w(G) \ldots$

$w(F \vee G) = \ldots w(F) \ldots w(G) \ldots$

Complexity considerations

The CNF and DNF of a formula of size n can have size 2^n

Can we do better? Yes, if we do not instist on $\equiv$.

Definition

Two formulas F and G are equisatisfiable if
F is satisfiable iff G is satisfiable.

Theorem

For every formula F of size n
there is an equisatisfiable CNF formula G of size $O(n)$.