Propositional Logic
Equivalences

Equivalence

Definition (Equivalence)

Two formulas F and G are (semantically) equivalent if $\mathcal{A}(F) = \mathcal{A}(G)$ for every assignment $\mathcal{A}$.

We write $F \equiv G$ to denote that F and G are equivalent.

Exercise

Which of the following equivalences hold?

$$
\begin{aligned}
(A \wedge (A \vee B)) &\equiv A \\
(A \wedge (B \vee C)) &\equiv ((A \wedge B) \vee C) \\
(A \rightarrow (B \rightarrow C)) &\equiv ((A \rightarrow B) \rightarrow C) \\
(A \rightarrow (B \rightarrow C)) &\equiv ((A \wedge B) \rightarrow C) \\
(A \rightarrow B) &\equiv (\neg A \vee B) \\
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Observation

The following connections hold:

$$\models F \to G \quad \text{iff} \quad F \models G$$
$$\models F \leftrightarrow G \quad \text{iff} \quad F \equiv G$$

NB: “iff” means “if and only if”

Reductions between problems (I)

- Validity to Unsatisfiability:

- Unsatisfiability to Validity:

- Validity to Consequence:

- Consequence to Validity:

- Validity to Equivalence:

- Equivalence to Validity:

Reductions between problems (I)

- Validity to Unsatisfiability:

$$F \text{ valid} \quad \text{iff} \quad \textbf{?} \text{ unsatisfiable}$$

- Unsatisfiability to Validity:

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- Validity to Unsatisfiability:

  $$F \text{ valid} \quad \text{iff} \quad \neg F \text{ unsatisfiable}$$

- Unsatisfiability to Validity:

  F unsatisfiable iff **?** valid

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Reductions between problems (I)

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$$F \text{ valid} \quad \text{iff} \quad \mathbf{?} \models \mathbf{?}$$

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$$F \text{ valid} \quad \text{iff} \quad \top \models F$$

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$$F \models G \quad \text{iff} \quad F \rightarrow G \text{ valid}$$

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- Consequence to Validity:

$$F \models G \quad \text{iff} \quad F \to G \text{ valid}$$

- Validity to Equivalence:

$$F \text{ valid} \quad \text{iff} \quad ? \equiv ?$$

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$$F \text{ valid} \quad \text{iff} \quad F \equiv \top$$

- Equivalence to Validity:

$$F \equiv G \quad \text{iff} \quad F \leftrightarrow G \text{ valid}$$

Properties of semantic equivalence

- Semantic equivalence is an equivalence relation between formulas.
- Semantic equivalence is closed under operators:

  If $F_1 \equiv F_2$ and $G_1 \equiv G_2$
  then $\neg F_1 \equiv \neg F_2$ and
  $(F_1 \circ G_1) \equiv (F_2 \circ G_2)$ for $\circ \in \{\vee, \wedge, \rightarrow, \leftrightarrow\}$

Equivalence relation + Closure under Operations
=
Congruence relation

Replacement theorem

Theorem

Let $F \equiv G$. Let H be a formula with an occurrence of F as a subformula. Let H' be the result of replacing an arbitrary occurrence of F in H by G. Then $H \equiv H'$.

Proof by induction on the structure of H.
We consider only the case $H = \neg H_0$.
Two cases: either $F = H$ or F is a subformula of H_0.

- $F = H$: Then $H' = G$ and thus $H = F \equiv G = H'$.
- F is a subformula of H_0.
  Let H_0' be the result of replacing F by G in H_0.
  IH: $H_0 \equiv H_0'$
  Thus $H = \neg H_0 \equiv \neg H_0' = H'$.

Equivalences (I)

Theorem

$$
\begin{array}{rcll}
(F \wedge F) & \equiv & F & \\
(F \vee F) & \equiv & F & \textit{(Idempotence)} \\
(F \wedge G) & \equiv & (G \wedge F) & \\
(F \vee G) & \equiv & (G \vee F) & \textit{(Commutativity)} \\
((F \wedge G) \wedge H) & \equiv & (F \wedge (G \wedge H)) & \\
((F \vee G) \vee H) & \equiv & (F \vee (G \vee H)) & \textit{(Associativity)} \\
(F \wedge (F \vee G)) & \equiv & F & \\
(F \vee (F \wedge G)) & \equiv & F & \textit{(Absorption)}
\end{array}
$$

Equivalences (II)

$$
\begin{array}{rcll}
(F \wedge (G \vee H)) & \equiv & ((F \wedge G) \vee (F \wedge H)) & \\
(F \vee (G \wedge H)) & \equiv & ((F \vee G) \wedge (F \vee H)) & \text{(Distributivity)} \\
\neg\neg F & \equiv & F & \text{(Double negation)} \\
\neg(F \wedge G) & \equiv & (\neg F \vee \neg G) & \\
\neg(F \vee G) & \equiv & (\neg F \wedge \neg G) & \text{(deMorgan's Laws)} \\
\neg\top & \equiv & \bot & \\
\neg\bot & \equiv & \top & \\
(\top \vee G) & \equiv & \top & \\
(\top \wedge G) & \equiv & G & \\
(\bot \vee G) & \equiv & G & \\
(\bot \wedge G) & \equiv & \bot &
\end{array}
$$

Warning

The symbols $\models$ and $\equiv$ are not operators
in the language of propositional logic
but part of the meta-language for talking about logic.

Examples:

$\mathcal{A} \models F$ and $F \equiv G$ are not propositional formulas.

$(\mathcal{A} \models F) \equiv G$ and $(F \equiv G) \leftrightarrow (G \equiv F)$ are nonsense.