Problem 1: Quiz (10 points)

For each of the following statements, decide whether they hold or not. If the statement holds, give a short proof. If the statement does not hold, give a counterexample.

a) For all propositional formulas F, G, H: if $F \wedge G \models H$ then $(F \rightarrow G) \rightarrow H$ is valid.

Solution

False. Take $F := A$, $G := B$, and $H := A$. Then

- $A \wedge B \models A$ holds, because every model of $A \wedge B$ is a model of A.
- $(A \rightarrow B) \rightarrow A$ is not valid, because the assignment $\{A \mapsto 0, B \mapsto 1\}$ makes $A \rightarrow B$ true and A false, and so it makes $(A \rightarrow B) \rightarrow A$ false.

Alternative solution: False. Take $F = G = H := \mathsf{false}$. Then $F \wedge G \equiv \mathsf{false}$ and $(F \rightarrow G) \rightarrow H \equiv \mathsf{false}$. So $F \wedge G \models H$, because false entails all formulas, and $(F \rightarrow G) \rightarrow H$ is not valid.

Remark: a) is an statement of the form "for all formulas F, G, H holds ......". A counterexample to a) is a set of three concrete formulas such that the statement does not hold.

b) The equivalence problem for DNF formulas is in P, even if P $\neq$ NP.

Solution

False. Given a formula F in CNF, we can convert it into a formula G in DNF of length linear in F such that $G \equiv \neg F$ (for that, start with F and push the negation inside using DeMorgan laws). We have: F is satisfiable iff G is valid iff $G \equiv \mathsf{true}$. Therefore, if the equivalence for DNF formulas is in P, even if P $\neq$ NP, then the satisfiability problem for CNF formulas is also in P, even if P $\neq$ NP. This contradicts that satisfiability of CNF formulas is NP-complete.

c) The set of all satisfiable formulas of first-order logic is a theory.

Solution

False. Let S be the set of satisfiable formulas. Assume S is a theory. Since $F := \exists x P(x)$ and $G := \neg \exists x P(x)$ are both satisfiable, both belong to S. Since $F, G \models F \wedge G$, and theories are closed under consequence, we have $F \wedge G \in S$. But $F \wedge G$ is unsatisfiable, so $F \wedge G \notin S$. Contradiction.

d) If $\mathcal{A}$ and $\mathcal{B}$ are structures over the same signature s.t. $U^{\mathcal{A}} \subseteq U^{\mathcal{B}}$, then $Th(\mathcal{A}) \subseteq Th(\mathcal{B})$.

Solution

False. Consider the signature containing only a predicate symbol P of arity 1. Let $\mathcal{A}$ and $\mathcal{B}$ be the structures given by:

- $U^{\mathcal{A}} = \{a\}$ and $P^{\mathcal{A}} = \{a\}$ (intuitively, $P(x)$ holds for $x = a$).
- $U^{\mathcal{B}} = \{a\}$ and $P^{\mathcal{B}} = \emptyset$ (intuitively, $P(x)$ does not hold for $x = a$).

We have $U^{\mathcal{A}} \subseteq U^{\mathcal{B}}$. Let $F = \exists x P(x)$. We have $\mathcal{A}(F) = 1$ and $\mathcal{B}(F) = 0$. Since *Th*($\mathcal{A}$) contains the formulas made true by $\mathcal{A}$, and similarly for $\mathcal{B}$, we get $F \in$ *Th*($\mathcal{A}$) and $F \notin$ *Th*($\mathcal{B}$). So *Th*($\mathcal{A}$) is not contained in *Th*($\mathcal{B}$).

e) Every sound and complete theory is decidable.

Solution

False. *Arithmetic* is sound and complete, but not decidable.

Problem 2: Horn formulas (5 points)

Let F be a propositional formula for which we have the following information (and no more):

(i) F is a Horn formula over the variables $\{A, B, C, D, E\}$.

(ii) Applying the satisfiability check for Horn formulas to F yields the satisfying assignment $\mathcal{A}$ given by: $\mathcal{A}(A) = 1$, $\mathcal{A}(B) = 0$, $\mathcal{A}(C) = 1$, $\mathcal{A}(D) = 0$, $\mathcal{A}(E) = 0$.

a) Which is the maximal number of satisfying assignments that F can have? Prove that every formula satisfying (i) and (ii) has at most this number, and give a formula satisfying (i) and (ii) with exactly that number of satisfying assignments.

Solution

The maximal number is 8. The satisfiability check for Horn formulas returns the unique satisfying assignment in which a minimal set of variables is set to 1. In other words, every assignment that satisfies F sets A and C to 1. So F can have at most 8 satisfying assignments. The formula $A \wedge (B \vee \neg B) \wedge C \wedge (D \to D) \wedge (E \to E)$ is a Horn formula with exactly 8 satisfying assignments (where we use the implication form of Horn-formulas). The formula $A \wedge B$ is also accepted.

b) Give a Horn formula G over the variables $\{A, B, C, D, E\}$ such that the assignment $\mathcal{A}$ of part (i) makes G true and G has exactly 3 satisfying assignments.

Solution

A possible formula is $G := A \wedge \neg B \wedge C \wedge (D \vee \neg E)$. Every satisfying assignment of G sets A and C to 1 and B to 0. This leaves 4 possible combinations for D, E. The conjunct $(D \vee \neg E)$ eliminates the combination where D is set to 0 and E is set to 1. The other 3 yield satisfying assignments.

Problem 3: Interpolants (3+6 bonus points)

Let F and G be propositional formulas such that $F \models G$. An *interpolant* between F and G is a formula H such that

- $F \models H \models G$ (meaning that $F \models H$ and $H \models G$), and
- H only contains variables than appear in both F and G.

Example: The formula y is an interpolant between $F = x \wedge y$ and $G = y \vee z$. On the contrary, $x \wedge y$ is not an interpolant because x does not appear in G.

a) Find an interpolant H between the formulas

$$F = (x \leftrightarrow \neg(y \wedge z)) \wedge (x \rightarrow (y \vee z))$$
$$G = y \vee (u \rightarrow (y \rightarrow z))$$

Prove that H indeed satisfies $F \models H \models G$.

Solution

$H := \mathsf{true}$ is an interpolant.

- $F \models \mathsf{true}$ holds for every formula F.
- We prove $\mathsf{true} \models G$. It suffices to show $G \equiv \mathsf{true}$ is valid. For this, observe that $G \equiv y \vee (u \rightarrow (\neg y \vee z)) \equiv y \vee (\neg u \vee (\neg y \vee z)) \equiv y \vee \neg y \vee \neg u \vee z \equiv \mathsf{true}$.

Alternative solution: $H := (y \vee z)$ is an interpolant.

- We prove $F \models H$, that is, every model of F is also model of H. Let $\mathcal{A}$ be a model of F. If $\mathcal{A}(x) = 0$, then $\mathcal{A}(y \vee z) = 1$ by the first conjunct of F, and if $\mathcal{A}(x) = 1$ then $\mathcal{A}(y \vee z) = 1$ by the second conjunct.
- We prove $H \models G$, that is, every model $\mathcal{A}$ of H is also model of G. Let $\mathcal{A}$ be a model of H, that is, $\mathcal{A}(y \vee z) = 1$. If $\mathcal{A}(y) = 1$ then $\mathcal{A}(G) = 1$ because of the first disjunct of G. Otherwise $\mathcal{A}(z) = 1$, which implies $\mathcal{A}(y \rightarrow z) = 1$ and so $\mathcal{A}(u \rightarrow (y \rightarrow z)) = 1$, which in turn implies $\mathcal{A}(G) = 1$.

b) **Bonus points.** Prove that every pair of formulas F, G such that $F \models G$ has at least one interpolant.

Solution

Given any set M of assignments to a set X of variables, there always exists a formula φ_M over X that is true exactly for the assignments of M, and no others. Let $X = \mathit{var}(F) \cap \mathit{var}(G)$. Given an assignment $\mathcal{A}$ to F, let $\mathcal{A}_X$ denote the projection of $\mathcal{A}$ onto X. Let $M = \{\mathcal{A}_X \mid \mathcal{A} \models F\}$, i.e., the projections onto X of all the assignments that satisfy F, and define $H := \varphi_M$.

We have:

1. $F \models H$. Let $\mathcal{A}$ satisfy $\mathcal{A}(F) = 1$. Then $\mathcal{A}_X(H) = 1$, and so $\mathcal{A}(H) = 1$ holds too.

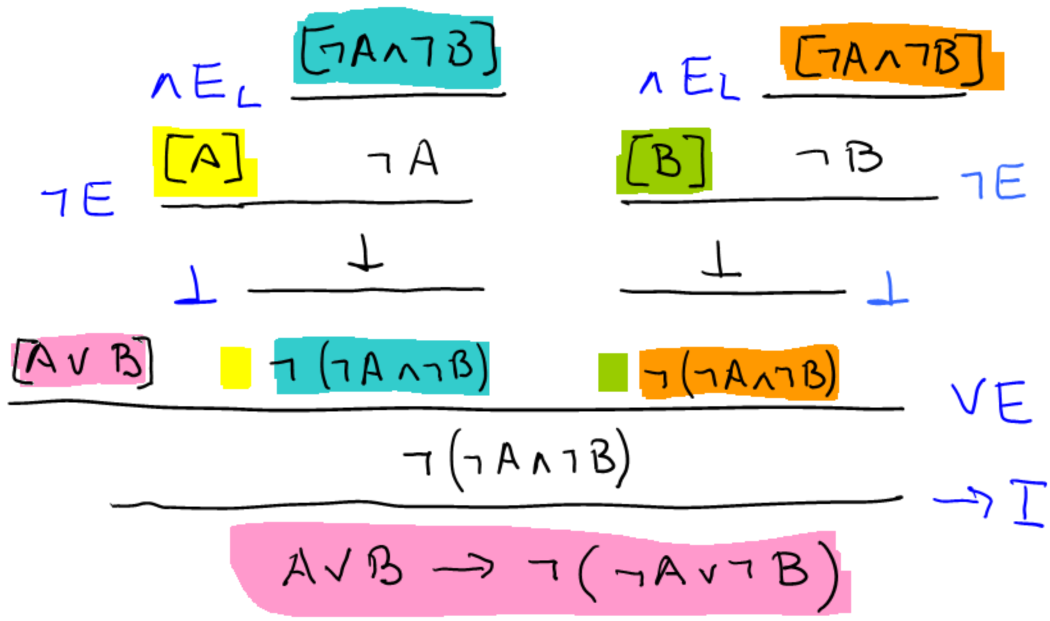
2. $H \models G$. Let $\mathcal{A}$ satisfy $\mathcal{A}(H) = 1$. By definition of H, there exists an assignment $\mathcal{B}$ of F such that $\mathcal{B}(F) = 1$ and $\mathcal{A} = \mathcal{B}_X$. Since $F \models G$, we have $\mathcal{B}(G) = 1$. Since $\mathcal{A}$ and $\mathcal{B}$ coincide on all variables of G, we get $\mathcal{A}(G) = 1$.

Problem 4: Natural deduction (3 points)

Prove the validity of the formula

$$(A \vee B) \rightarrow \neg(\neg A \wedge \neg B)$$

using natural deduction.

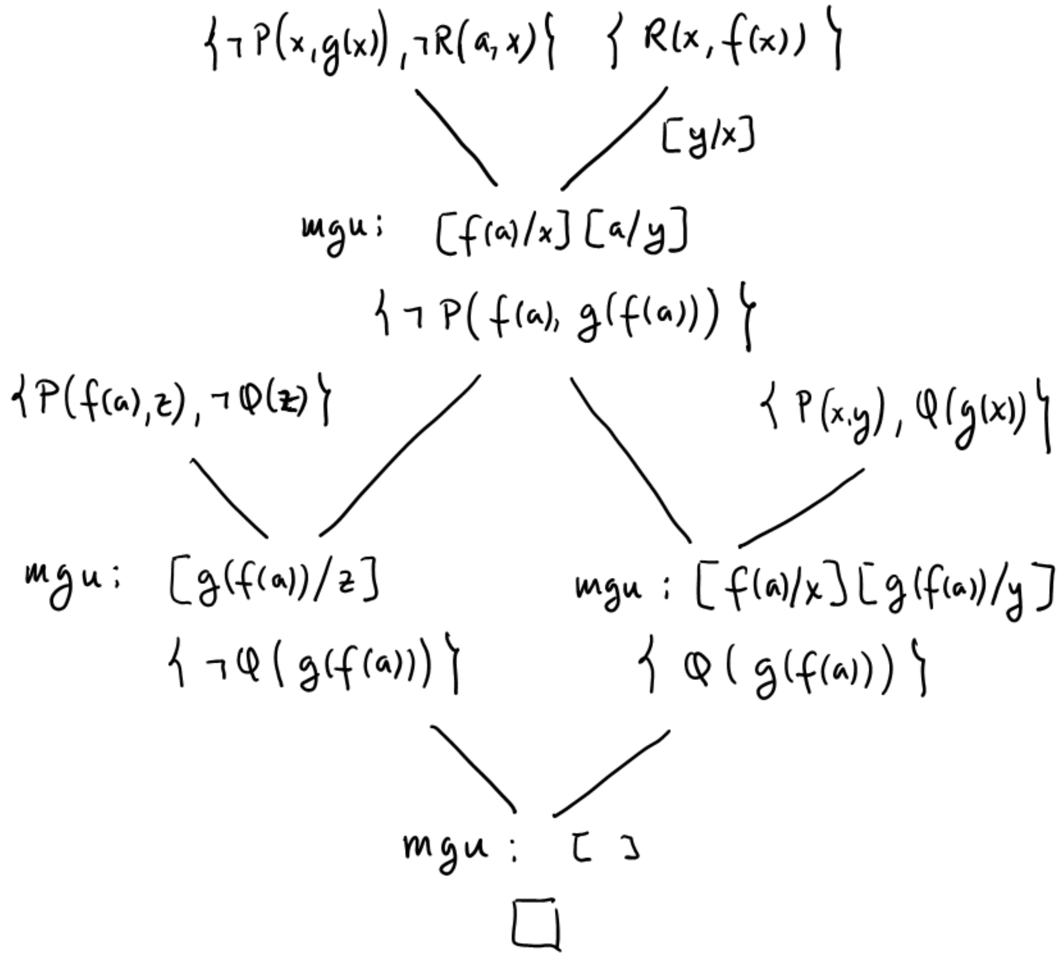


Problem 5: Resolution (3 points)

Show that the set of clauses below is unsatisfiable using first-order logic resolution. In the clauses, x, y, z are variables, f and g are function symbols of arity 1, and a is a constant.

$$\{P(f(a), z), \neg Q(z)\} \quad \{P(x, y), Q(g(x))\} \quad \{\neg P(x, g(x)), \neg R(a, x)\} \quad \{R(x, f(x))\}$$

For each resolution step give all the substitutions and the most general unifier used in the step.



Problem 6: Modelling (4 points)

Let R be a binary predicate symbol.

a) Give a formula of first-order logic F with signature $\{R\}$ such that $\mathcal{A}$ is a model of F iff $R^{\mathcal{A}}$ is an equivalence relation with exactly two equivalence classes.

Solution

We can take F as the conjunction of:

- $\forall x\ R(x,x)$ ($R^{\mathcal{A}}$ is reflexive).
- $\forall x \forall y\ (R(x,y) \rightarrow R(y,x))$ ($R^{\mathcal{A}}$ is symmetric).
- $\forall x \forall y \forall z\ \big((R(x,y) \wedge R(y,z)) \rightarrow R(x,z)\big)$ ($R^{\mathcal{A}}$ is transitive).
- $\exists x \exists y\ \neg R(x,y)$ ($R^{\mathcal{A}}$ has at least two equivalence classes).
- $\forall x \forall y \forall z\ (R(x,y) \vee R(x,z) \vee R(y,z))$ ($R^{\mathcal{A}}$ has at most two equivalence classes).

The last two formulas can be replaced by $\exists x \exists y \big(\neg R(x,y) \wedge \forall z((R(x,z) \vee R(y,z))\big)$

b) Give a formula of first-order logic F with signature $\{R\}$ such that $\mathcal{A}$ is a model of F iff $R^{\mathcal{A}}$ is a total order with a maximal element. (The order $\leq$ on the natural numbers is an example of a total order that does not have a maximal element.)

Solution

We can take F as the conjunction of:

- $\forall x \forall y\ R(x,x)$ ($R^{\mathcal{A}}$ is reflexive).
- $\forall x \forall y\ \big(R(x,y) \rightarrow (\neg R(y,x) \vee x = y))\big)$ ($R^{\mathcal{A}}$ is antisymmetric).
- $\forall x \forall y \forall z\ \big((R(x,y) \wedge R(y,z)) \rightarrow R(x,z)\big)$ ($R^{\mathcal{A}}$ is transitive).
- $\forall x \forall y\ (R(x,y) \vee R(y,x))$ ($R^{\mathcal{A}}$ is total).
- $\exists x \forall y\ R(y,x)$ ($R^{\mathcal{A}}$ has a maximal element).

We also accept as solution the variant saying that without equality one can only specify irreflexive order relations, that is, orders like $<$ (proof: if one could specify that $R(x,y)$ is a reflexive order relation, then one could also specify equality as $R(x,y) \wedge R(y,x)$, which as seen in the lectures is not possible) and giving the conjunction of:

- $\forall x \forall y\ \neg R(x,x)$ ($R^{\mathcal{A}}$ is irreflexive).
- $\forall x \forall y\ (R(x,y) \rightarrow \neg R(y,x))$ ($R^{\mathcal{A}}$ is asymmetric).
- $\forall x \forall y \forall z\ \big((R(x,y) \wedge R(y,z)) \rightarrow R(x,z)\big)$ ($R^{\mathcal{A}}$ is transitive).
- $\forall x \forall y\ (R(x,y) \vee R(y,x))$ ($R^{\mathcal{A}}$ is total).
- $\exists x \forall y\ R(y,x)$ ($R^{\mathcal{A}}$ has a maximal element).

Problem 7: Herbrand theory (7 points)

Let $F = \exists x(P(x) \rightarrow \forall x P(x))$.

a) Transform F into a formula G in Skolem normal form such that G does not contain any function symbols of arity 1 or more. Give the intermediate steps of the transformation.

Solution

$$\begin{aligned} & \exists x(P(x) \rightarrow \forall x P(x)) \\ \equiv \quad & \exists x(P(x) \rightarrow \forall y P(y)) \\ \equiv \quad & \exists x(\neg P(x) \vee \forall y P(y)) \\ \equiv \quad & \exists x \forall y(\neg P(x) \vee P(y)) \\ \equiv_s \quad & \forall y(\neg P(a) \vee P(y)) \end{aligned}$$

where $\equiv_s$ means equisatisfiable.

b) Enumerate all Herbrand structures of G.

Solution

The Herbrand universe is $\{a\}$.

There are two Herbrand structures $\mathcal{A}_1$ and $\mathcal{A}_2$ with universe $\{a\}$ and $P_1^{\mathcal{A}} = \emptyset$ and $P_2^{\mathcal{A}} = \{a\}$.

c) Is F valid, satisfiable, or unsatisfiable? Prove your answer.

Solution

F is valid. A possible way to show it is:

$$\begin{aligned} & \exists x(P(x) \rightarrow \forall x P(x)) \\ \equiv \quad & \exists x(P(x) \rightarrow \forall y P(y)) \\ \equiv \quad & \exists x(\neg P(x) \vee \forall y P(y)) \\ \equiv \quad & \exists x \neg P(x) \vee \forall y P(y) \qquad (x \text{ does not appear in } \forall y P(y)) \\ \equiv \quad & \neg \forall x P(x) \vee \forall y P(y) \\ \equiv \quad & \neg \forall x P(x) \vee \forall x P(x) \\ \equiv \quad & \text{true} \end{aligned}$$

Notice that an answer like "F is valid because all Herbrand structures of G are models" is not correct. A formula is satisfiable iff some Herbrand structure is a model, but there exist non-valid formulas whose Herbrand structures are all models.

Problem 8: Linear arithmetic (5 points)

Consider the formula

$$\varphi(x) = \forall y \exists z \; \big(3y + 1 \leq x \;\vee\; 2y \leq 5x \;\vee\; (z \leq y + 1 \;\wedge\; 2x \leq z + 1)\big)$$

of linear arithmetic over the rational numbers. Apply the quantifier elimination algorithm of the course to compute all values of x for which the formula holds. Give the formulas obtained after eliminating each quantifier, and any other intermediate formulas needed to understand how you applied the algorithm. Underline the final result.

Solution

$$\begin{aligned}
& \forall y \exists z \; \big(3y + 1 \leq x \;\vee\; 2y \leq 5x \;\vee\; (z \leq y + 1 \;\wedge\; 2x \leq z + 1)\big) \\
\equiv\; & \forall y \; \big(3y + 1 \leq x \;\vee\; 2y \leq 5x \;\vee\; \exists z (z \leq y + 1 \;\wedge\; 2x \leq z + 1)\big) \\
\equiv\; & \forall y \; \big(3y + 1 \leq x \;\vee\; 2y \leq 5x \;\vee\; \exists z (2x - 1 \leq z \wedge z \leq y + 1\big) \\
\equiv\; & \forall y \; \big(3y + 1 \leq x \;\vee\; 2y \leq 5x \;\vee\; 2x - 1 \leq y + 1\big) && (z \text{ eliminated}) \\
\equiv\; & \neg \exists y \; \big(3y + 1 > x \;\wedge\; 2y > 5x \;\wedge\; 2x - 1 > y + 1\big) \\
\equiv\; & \neg \exists y \; \big(\tfrac{1}{3}(x - 1) < y \;\wedge\; \tfrac{5}{2}x < y \;\wedge\; y < 2x - 2\big) \\
\equiv\; & \neg \big(\tfrac{1}{3}(x - 1) < 2x - 2 \;\wedge\; \tfrac{5}{2}x < 2x - 2\big) && (y \text{ eliminated}) \\
\equiv\; & \neg \big(1 < x \;\wedge\; \tfrac{5}{2}x < 2x - 2\big) \\
\equiv\; & \neg \big(1 < x \;\wedge\; x < -4\big) \\
\equiv\; & \underline{x \leq 1 \;\vee\; x \geq -4}
\end{aligned}$$

Since every rational number is either below 1 or above -4, the formula holds for every value of x.