Exercise sheet: Quantifier Elimination

Exercise 1: Fourier-Motzkin Elimination

Apply the Fourier–Motzkin Elimination to check the following sentences:

1. $\exists x \exists y (2 \cdot x + 3 \cdot y = 7 \land x < y \land 0 < x)$
2. $\exists x \exists y (3 \cdot x + 3 \cdot y < 8 \land 8 < 3 \cdot x + 2 \cdot y)$

Use $\iff$ if two formulas are logically equivalent and $\iff_{R_+}$ if the equivalence requires the theory R_+.

Solution

$$\begin{aligned}
& \exists x \exists y (2 \cdot x + 3 \cdot y = 7 \land x < y \land 0 < x) \\
\iff \quad & \exists x \left(\exists y \left(2 \cdot x + 3 \cdot y = 7 \land x < y\right) \land 0 < x\right) \\
\iff_{R_+} \quad & \exists x \left(\exists y \left(y = \frac{7}{3} - \frac{2}{3} \cdot x \land x < y\right) \land 0 < x\right) \\
\iff_{R_+} \quad & \exists x \left(x < \frac{7}{3} - \frac{2}{3} \cdot x \land 0 < x\right) \\
\iff_{R_+} \quad & \exists x \left(x < \frac{7}{5} \land 0 < x\right) \\
\iff_{R_+} \quad & 0 < \frac{7}{5} \\
\iff_{R_+} \quad & \top \qquad \text{(optional step; not part of QEP)}
\end{aligned}$$

$$\begin{aligned}
& \exists x \exists y (3 \cdot x + 3 \cdot y < 8 \land 8 < 3 \cdot x + 2 \cdot y) \\
\iff_{R_+} \quad & \exists x \exists y \left(y < \frac{8}{3} - x \land 4 - \frac{3}{2} \cdot x < y\right) \\
\iff_{R_+} \quad & \exists x \left(4 - \frac{3}{2} \cdot x < \frac{8}{3} - x\right) \\
\iff_{R_+} \quad & \exists x \left(\frac{8}{3} < x\right) \\
\iff_{R_+} \quad & \top
\end{aligned}$$

Exercise 2: Ferrante-Rackoff Elimination

Apply the Ferrante–Rackoff Elimination to check the validity of the following sentence:

$$\exists x (\exists y (x = 2 \cdot y) \to (2 \cdot x \geq 0 \lor 3 \cdot x < 2))$$

$$
\begin{aligned}
& \exists x(\exists y(x = 2 \cdot y) \to (2 \cdot x \geq 0 \vee 3 \cdot x < 2)) \\
\Longleftrightarrow_{R_+}\ & \exists x(\exists y(y = \frac{1}{2}x) \to (2 \cdot x \geq 0 \vee 3 \cdot x < 2)) \\
\Longleftrightarrow_{R_+}\ & \exists x\left(\bot \vee \bot \vee \frac{1}{2}x = \frac{1}{2}x \vee \bot \to (2 \cdot x \geq 0 \vee 3 \cdot x < 2)\right) \\
\Longleftrightarrow\ & \exists x\left(\left(\top \wedge \top \wedge \neg\left(\frac{1}{2}x = \frac{1}{2}x\right) \wedge \top\right) \vee (2 \cdot x \geq 0 \vee 3 \cdot x < 2)\right) \\
\Longleftrightarrow_{R_+}\ & \exists x\left(\left(\top \wedge \top \wedge \left(\frac{1}{2}x < \frac{1}{2}x\right) \wedge \top\right) \vee (2 \cdot x \geq 0 \vee 3 \cdot x < 2)\right) \\
\Longleftrightarrow_{R_+}\ & \exists x\left((\top \wedge \top \wedge \bot \wedge \top) \vee \left(0 < x \vee x = 0 \vee x < \frac{2}{3}\right)\right) \\
\Longleftrightarrow_{R_+}\ & \bigvee_{t \in \{-\infty, \infty, 0, 1/3\}} \left((\top \wedge \top \wedge \bot \wedge \top) \vee \left(0 < x \vee x = 0 \vee x < \frac{2}{3}\right)\right)[t/x] \\
\Longleftrightarrow\ & (\cdots \vee (\bot \vee \bot \vee \top)) \vee (\cdots \vee (\top \vee \bot \vee \bot)) \vee \left(\cdots \vee \left(0 < 0 \vee 0 = 0 \vee 0 < \frac{2}{3}\right)\right) \\
& \vee \left(\cdots \vee \left(0 < \frac{1}{3} \vee \frac{1}{3} = 0 \vee \frac{1}{3} < \frac{2}{3}\right)\right) \\
\Longleftrightarrow\ & \top \qquad \text{(optional step; not part of QEP)}
\end{aligned}
$$

Exercise 3: Presburger Arithmetic

Using quantifier elimination check whether the following sentence belongs to Presburger arithmetic.

$$\forall x \exists y\big((x < 2y + 1 \wedge 2y < x + 1) \vee (x < 2y + 2 \wedge 2y < x)\big)$$

Solution

Note that

$$
\begin{aligned}
& \forall x \exists y\big((x < 2y + 1 \wedge 2y < x + 1) \vee (x < 2y + 2 \wedge 2y < x)\big) \\
\equiv & \forall x\big(\exists y(x \leq 2y \wedge 2y \leq x) \vee \exists y(x - 1 \leq 2y \wedge 2y \leq x - 1)\big)
\end{aligned}
$$

Let $F_1 = \exists y(x \leq 2y \wedge 2y \leq x)$ and $F_2 = \exists y(x - 1 \leq 2y \wedge 2y \leq x - 1)$. We first eliminate quantifiers from these two subformulas.

For eliminating the quantifier $\exists y$ from F_1, we proceed in two steps. First, we need to set all the coefficients of y to either 1 or -1. To this end, performing the first step of the quantifier elimination procedures yields the following equivalent formula G_1.

$$G_1 = \exists y(x \leq y \ \wedge \ y \leq x \ \wedge \ 2 \mid y)$$

We can now perform the second step of the quantifier elimination procedure on G_1. To this end, note that $A_L = \{0 \leq y - x\}, A_U = \{0 \leq -y + x\}, L = \{x\}, U = \{x\}, D =$

$\{2 \mid y\}$ and so the performing the second step of the quantifier elimination procedure on G_1 yields the following equivalent formula H_1.

$$\begin{aligned} H_1 &= ((x \leq x) \wedge (x \leq x) \wedge (2 \mid x)) \vee ((x \leq x+1) \wedge (x+1 \leq x) \wedge (2 \mid x+1)) \\ &\equiv 2 \mid x \end{aligned}$$

Similarly, from F_2 we obtain $H_2 = 2 \mid x - 1$. Consequently, the initial formula is equivalent to $H = \forall x((2 \mid x) \vee (2 \mid x-1))$. Now observe that $\neg(m \mid n) \equiv \bigvee_{1 \leq i < m} m \mid n + i$. Hence,

$$\begin{aligned} &\forall x((2 \mid x) \vee (2 \mid x-1)) \\ \equiv &\neg \exists x(\neg(2 \mid x) \wedge \neg(2 \mid x-1)) \\ \equiv &\neg \exists x((2 \mid x+1) \wedge (2 \mid x)) \end{aligned}$$

We now eliminate x from $(2 \mid x+1) \wedge (2 \mid x)$. Note that we do not need to apply the first step of quantifier elimination, since all the coefficients of x are already either 1 or -1. Performing the second step allows us to derive the following.

$$\begin{aligned} &\exists x\big((2 \mid x+1) \wedge (2 \mid x)\big) \\ \equiv &((2 \mid 0+1) \wedge (2 \mid 0)) \vee ((2 \mid 1+1) \wedge (2 \mid 1)) \\ \equiv &((2 \mid 1) \wedge (2 \mid 0)) \vee ((2 \mid 2) \wedge (2 \mid 1)) \\ \equiv &\mathit{false}. \end{aligned}$$

Finally, $\neg \mathit{false} \equiv \mathit{true}$, which shows that the initial formula is true in Presburger arithmetic.

Exercise 4: Completeness

Which of the following theories are complete? Justify your answers.

1. Presburger arithmetic,
2. Theory of linear orders,
3. Theory of dense linear orders,
4. Group theory.

Solution

1. Presburger arithmetic is complete since it is defined as a theory of a structure. For every formula, this structure is either a model or it is a model for its negation, and thus, for every F, either F or $\neg F$ is in Presburger arithmetic, which makes it complete.
2. The theory of linear orders is not complete, since neither the formula $\forall x \exists y (y < x)$ nor its negation belong to the theory.

3. The theory of dense linear orders is also not complete, and the same sentence proves that as in the previous case.

4. The group theory is not complete, as neither the formula $\forall x \forall y (x \cdot y = y \cdot x)$ nor its negation belong to it. There exist both abelian and non-abelian groups.