Exercise sheet: Propositional Proof Theory

Exercise 1: Natural Deduction

Prove the following formula using natural deduction.

$$\neg(\forall x(\exists y(\neg P(x) \land P(y))))$$

Solution

$$\dfrac{\dfrac{\dfrac{[\forall x(\exists y(\neg P(x) \land P(y)))]_1}{\exists y(\neg P(x_1) \land P(y))}\,\forall\mathsf{E} \qquad \dfrac{\dfrac{[\neg P(x_1) \land P(y_1)]_2}{P(y_1)}\,\land\mathsf{E}_2 \qquad \dfrac{\dfrac{[\forall x(\exists y(\neg P(x) \land P(y)))]_1}{\exists y(\neg P(y_1) \land P(y))}\,\forall\mathsf{E} \qquad \dfrac{[\neg P(y_1) \land P(y_2)]_3}{\neg P(y_1)}\,\land\mathsf{E}_1}{\neg P(y_1)}\,\exists\mathsf{E}(3)}{\bot}\,\neg\mathsf{E}}{\bot}\,\exists\mathsf{E}(2)}{\neg(\forall x(\exists y(\neg P(x) \land P(y))))}\,\neg\mathsf{I}(1)$$

Exercise 2: Sequent Calculus

Prove the following formulae in sequent calculus:

1. $\neg\exists x P(x) \to \forall x \neg P(x)$
2. $(\forall x(P \lor Q(x))) \to (P \lor \forall x Q(x))$

Solution

1.
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P(y) \Rightarrow P(y), \exists x P(x)}\,\mathsf{Ax}}{\Rightarrow P(y), \exists x P(x), \neg P(y)}\,\neg\mathsf{R}}{\Rightarrow \exists x P(x), \neg P(y)}\,\exists\mathsf{R}}{\Rightarrow \exists x P(x), \forall x \neg P(x)}\,\forall\mathsf{R}}{\neg\exists x P(x) \Rightarrow \forall x \neg P(x)}\,\neg\mathsf{L}}{\Rightarrow \neg\exists x P(x) \to \forall x \neg P(x)}\,\to\mathsf{R}$$

2.
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\forall x(P \lor Q(x)), P \Rightarrow P, Q(x)}\,\mathsf{Ax} \qquad \dfrac{}{\forall x(P \lor Q(x)), Q(x) \Rightarrow P, Q(x)}\,\mathsf{Ax}}{\forall x(P \lor Q(x)), (P \lor Q(x)) \Rightarrow P, Q(x)}\,\lor\mathsf{L}}{\forall x(P \lor Q(x)) \Rightarrow P, Q(x)}\,\forall\mathsf{L}}{\forall x(P \lor Q(x)) \Rightarrow P, \forall x Q(x)}\,\forall\mathsf{R}}{\forall x(P \lor Q(x)) \Rightarrow P \lor \forall x Q(x)}\,\lor\mathsf{R}}{\Rightarrow \forall x(P \lor Q(x)) \to P \lor \forall x Q(x)}\,\to\mathsf{R}$$

Exercise 3: Natural Deduction can Simulate Sequent Calculus II

In exercise 6.2 we proved that if $\Gamma \vdash_G \Delta$ then $\Gamma \vdash_N \bigvee \Delta$ for formulae in propositional logic. Augment your proof by the new cases for FOL.

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Case 1: $\dfrac{F[t/x], \forall x\ F, \Gamma \vdash_G \Delta}{\forall x\ F, \Gamma \Rightarrow \Delta}\ \forall\mathsf{L}$

By IH we have $T : F[t/x], \forall x\ F, \Gamma \vdash_N \bigvee \Delta$

We obtain our proof by replacing every occurance of the open assumption $F[t/x]$ in T by the prooftree $\dfrac{\forall x\ F}{F[t/x]}\ \forall\mathsf{E}$.

Case 2: $\dfrac{\Gamma \vdash_G F[y/x], \Delta}{\Gamma \Rightarrow \forall x\ F, \Delta}\ \forall\mathsf{R}$ Where $y \notin \mathsf{fv}(\Gamma, \forall x\ F, \Delta)$

By IH we have $T\colon \Gamma \vdash_N F[y/x] \vee \bigvee \Delta$

$$\dfrac{T \qquad \dfrac{\dfrac{[F[y/x]]_1}{\forall x\ F}\ \forall\mathsf{I}}{\forall x\ F \vee \bigvee \Delta}\ \vee\mathsf{I}_1 \qquad \dfrac{[\bigvee \Delta]_1}{\forall x\ F \vee \bigvee \Delta}\ \vee\mathsf{I}_2}{\forall x\ F \vee \bigvee \Delta}\ \vee\mathsf{E}(1)$$

Case 3: $\dfrac{F[y/x], \Gamma \vdash_G \Delta}{\exists x\ F, \Gamma \Rightarrow \Delta}\ \exists\mathsf{L}$ where $y \notin \mathsf{fv}(F, \Gamma, \Delta)$

By IH we have $T\colon F[y/x], \Gamma \vdash_N \bigvee \Delta$

$$\dfrac{\exists x\ F \qquad T}{\bigvee \Delta}\ \exists\mathsf{E}(1)$$

Case 4: $\dfrac{\Gamma \vdash_G F[t/x], \exists x\ F, \Delta}{\Gamma \Rightarrow \exists x\ F, \Delta}\ \exists\mathsf{R}$

By IH we have $T\colon \Gamma \vdash_N F[t/x] \vee \exists x\ F \vee \bigvee \Delta$

$$\dfrac{T \qquad \dfrac{\dfrac{[F[t/x]]_1}{\exists x\ F}\ \exists\mathsf{I}}{\exists x\ F \vee \bigvee \Delta}\ \vee\mathsf{I}_1 \qquad [\exists x\ F \vee \bigvee \Delta]_1}{\exists x\ F \vee \bigvee \Delta}\ \vee\mathsf{E}(1)$$

Exercise 4: Counterexamples from Sequent Calculus

Consider the statement $\forall x(P(x) \to \neg P(f(x)))$.

1. What happens when trying to prove the validity of this formula in sequent calculus?
2. How can we derive a countermodel from the proof tree?
3. Is there a smaller countermodel?

Solution

The proof tree gets stuck:

$$\dfrac{\dfrac{\dfrac{P(y), P(f(y)) \Rightarrow}{P(y) \Rightarrow \neg P(f(y))}\ \neg\mathsf{R}}{\Rightarrow P(y) \to \neg P(f(y))}\ \to \mathsf{R}}{\Rightarrow \forall x(P(x) \to \neg P(f(x)))}\ \forall\mathsf{R}$$

As in the lecture, we can create a countermodel $\mathcal{A}$: Let $U_{\mathcal{A}}$ be the set of all terms over $y, f(\cdot)$, set $y^{\mathcal{A}} \coloneqq y$, $f^{\mathcal{A}}(t) \coloneqq f(t^{\mathcal{A}})$, and $P^{\mathcal{A}} \coloneqq \{y, f(y)\}$. Then $\mathcal{A} \models P(y)$ and $\mathcal{A} \models P(f(y))$ and hence $\mathcal{A} \not\models \forall x\,(P(x) \to \neg P(f(x)))$. Note that $\mathcal{A}$ is infinite, but there are countermodels with just two elements $\{a, b\}$: Set $f(a) \coloneqq b$, $f(b) \coloneqq b$, $P(a)$ and $P(b)$. Then $P(a)$ and $P(f(a)) = P(b)$.