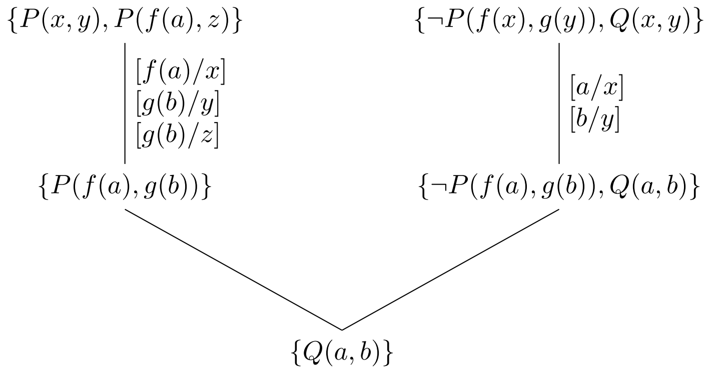
Exercise sheet: Resolution and Equality

Exercise 1: Lifting Lemma

Consider the following resolution:



Follow the proof of the Lifting Lemma and find out which (predicate logic) resolution step is constructed from this.

Exercise 2: Simulating Equality

(a) Show that the following formula has a Herbrand Model:

$$F \coloneqq \forall x \forall y (f(x) = f(y) \rightarrow x = y)$$

(b) Construct $G \coloneqq E_F \wedge F[Eq/=]$ as described in the lecture slides.

(c) Give a model of G that is not a model of F

Exercise 3: Equality in Herbrand's theorem

Let $\mathcal{A}$ be a structure with signature τ. Moreover, let τ_f, τ_R be a partition of τ such that τ_f only contains function symbols and τ_R only predicate symbols. For the rest of this exercise we assume that there exists at least one constant symbol $c \in \tau_f$. Furthermore, we consider first-order logic with equality in this exercise. Let $\mathcal{U}$ be the ground terms constructed from τ_f.

1. Prove that $\sim_{\mathcal{A}} \subseteq \mathcal{U} \times \mathcal{U}$ with

$$t_1 \sim_{\mathcal{A}} t_2 \text{ iff } \mathcal{A} \models t_1 = t_2$$

is an equivalence relation. As usual we use $[t]_{\sim_{\mathcal{A}}} = \{t' \in \mathcal{U} \mid t \sim_{\mathcal{A}} t'\}$.

2. Let $P \in \tau_R$ be a predicate symbol with arity k. Show that for all $t_1, \ldots, t_k, t'_1, \ldots, t'_k \in \mathcal{U}$ with $t_i \sim_{\mathcal{A}} t'_i$ for all $1 \in \{1, \ldots, k\}$ holds

$$\mathcal{A} \models P(t_1, \ldots, t_k) \text{ iff } \mathcal{A} \models P(t'_1, \ldots, t'_k).$$

3. Let φ be a satisfiable closed formula in Skolem normal form over the signature τ and $\mathcal{A} \models \varphi$. Prove that there exists a model of τ with universe $\mathcal{U}_{/\sim_{\mathcal{A}}} = \{[t]_{\sim_{\mathcal{A}}} : t \in \mathcal{U}\}$.

   Conclude that Herbrand's theorem can be generalized to first-order with equality.

4. Apply your generalization from above to the sentence you gave for Exercise 2 in the last exercise sheet.

5. Consider the following Formula:

$$F := \forall x(f(f(x)) = x)$$

   a) Give two models $\mathcal{A}$ and $\mathcal{B}$ for F such $\sim_{\mathcal{A}}$ and $\sim_{\mathcal{B}}$ differ.

   b) Give the sets $\mathcal{U}_{/\sim_{\mathcal{A}}}$ and $\mathcal{U}_{/\sim_{\mathcal{B}}}$