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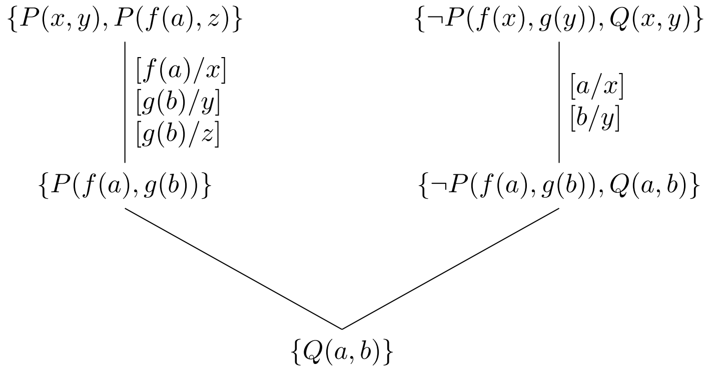
Exercise sheet: Resolution and Equality

Exercise 1: Lifting Lemma

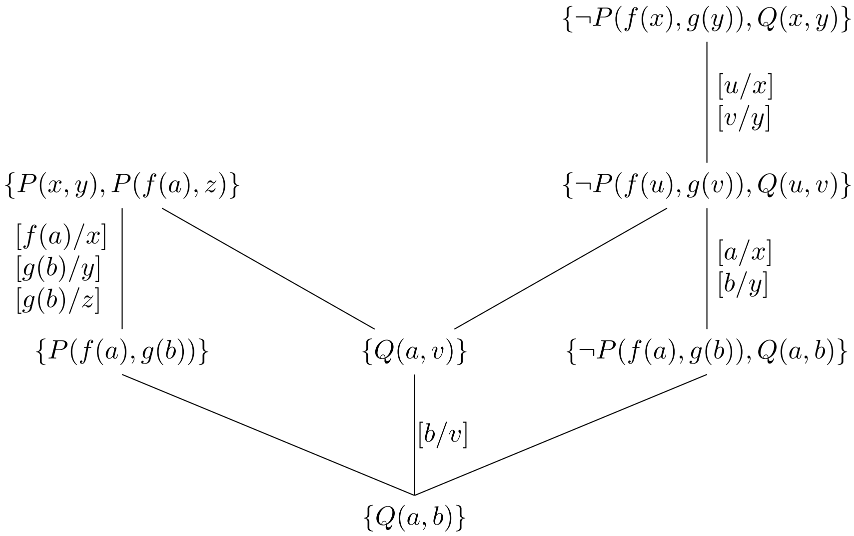
Consider the following resolution:



Follow the proof of the Lifting Lemma and find out which (predicate logic) resolution step is constructed from this.

Solution

The missing predicate resolution step can be depicted as follows:



Exercise 2: Simulating Equality

(a) Show that the following formula has a Herbrand Model:

$$F := \forall x \forall y (f(x) = f(y) \rightarrow x = y)$$

(b) Construct $G := E_F \land F[Eq/=]$ as described in the lecture slides.

(c) Give a model of G that is not a model of F

Solution

(a) In the Herbrand structure $\mathcal{H}$, the function $f^{\mathcal{H}}$ maps a term t to the term $f(t)$. Since $f(t) = f(t')$ iff $t = t'$, it follows that $f^{\mathcal{H}}$ is injective and $\mathcal{H} \models F$.

(b)

$$\begin{aligned} E_F &= \forall x. Eq(x, x) \\ &\quad \land \forall x \forall y. Eq(x, y) \rightarrow Eq(y, x) \\ &\quad \land \forall x \forall y \forall z. Eq(x, y) \rightarrow Eq(y, z) \rightarrow Eq(x, z) \\ &\quad \land \forall x \forall y. Eq(x, y) \rightarrow Eq(f(y), f(x)) \\ F[Eq/=] &= \forall x \forall y. (Eq(f(x), f(y)) \rightarrow Eq(x, y)) \end{aligned}$$

(c) We construct $\mathcal{A}$ over the universe $\mathcal{U}^{\mathcal{A}} = \mathbb{Z}$ and choose:

$$\begin{aligned} f^{\mathcal{H}}(x) &:= x^2 \\ Eq^{\mathcal{H}}(x, y) &:\Leftrightarrow |x| = |y| \end{aligned}$$

$\mathcal{A}$ is a model of G since $|x^2| = |y^2|$ iff $|x| = |y|$. But since it does not imply $x = y$, $\mathcal{A}$ is not a model of F.

Exercise 3: Equality in Herbrand's theorem

Let $\mathcal{A}$ be a structure with signature τ. Moreover, let τ_f, τ_R be a partition of τ such that τ_f only contains function symbols and τ_R only predicate symbols. For the rest of this exercise we assume that there exists at least one constant symbol $c \in \tau_f$. Furthermore, we consider first-order logic with equality in this exercise. Let $\mathcal{U}$ be the ground terms constructed from τ_f.

1. Prove that $\sim_{\mathcal{A}} \subseteq \mathcal{U} \times \mathcal{U}$ with

$$t_1 \sim_{\mathcal{A}} t_2 \text{ iff } \mathcal{A} \models t_1 = t_2$$

is an equivalence relation. As usual we use $[t]_{\sim_{\mathcal{A}}} = \{t' \in \mathcal{U} \mid t \sim_{\mathcal{A}} t'\}$.

2. Let $P \in \tau_R$ be a predicate symbol with arity k. Show that for all $t_1, \dots, t_k, t'_1, \dots, t'_k \in \mathcal{U}$ with $t_i \sim_{\mathcal{A}} t'_i$ for all $1 \in \{1, \dots, k\}$ holds

$$\mathcal{A} \models P(t_1, \dots, t_k) \text{ iff } \mathcal{A} \models P(t'_1, \dots, t'_k).$$

3. Let φ be a satisfiable closed formula in Skolem normal form over the signature τ and $\mathcal{A} \models \varphi$. Prove that there exists a model of τ with universe $\mathcal{U}_{/\sim_{\mathcal{A}}} = \{[t]_{\sim_{\mathcal{A}}} : t \in \mathcal{U}\}$.

   Conclude that Herbrand's theorem can be generalized to first-order with equality.

4. Apply your generalization from above to the sentence you gave for Exercise 2 in the last exercise sheet.

5. Consider the following Formula:

$$F := \forall x(f(f(x)) = x)$$

   a) Give two models $\mathcal{A}$ and $\mathcal{B}$ for F such $\sim_{\mathcal{A}}$ and $\sim_{\mathcal{B}}$ differ.

   b) Give the sets $\mathcal{U}_{/\sim_{\mathcal{A}}}$ and $\mathcal{U}_{/\sim_{\mathcal{B}}}$

Solution

1. Establishing that $\sim_{\mathcal{A}}$ is an equivalence relation is an immediate consequence from the fact that $=$ is an equivalence relation. We observe that if $t \sim_{\mathcal{A}} t'$ we get $\mathcal{A} \models t = t'$. Since $=$ is always interpreted by the equality relation we know that $t^{\mathcal{A}} = t'^{\mathcal{A}}$.

   Reflexive Trivially, $t^{\mathcal{A}} = t^{\mathcal{A}}$ and, therefore, $\mathcal{A} \models t = t$ which is the definition of $t \sim_{\mathcal{A}} t$.

   Symmetric If $t^{\mathcal{A}} = t'^{\mathcal{A}}$ then $t^{\mathcal{A}} = t'^{\mathcal{A}}$ and, consequently, $t \sim_{\mathcal{A}} t'$ implies $t' \sim_{\mathcal{A}} t$.

   Transitive Let $t_1 \sim_{\mathcal{A}} t_2$ and $t_2 \sim_{\mathcal{A}} t_3$. We know then that $t_1^{\mathcal{A}} = t_2^{\mathcal{A}}$ and $t_2^{\mathcal{A}} = t_3^{\mathcal{A}}$. Note that $t_1^{\mathcal{A}} = t_3^{\mathcal{A}}$ and, therefore, $\mathcal{A} \models t_1 = t_3$ which imples the desired $t_1 \sim_{\mathcal{A}} t_3$.

2. Due to the symmetry of the statement it suffices to only show one direction. Thus, we assume $\mathcal{A} \models P(t_1, \dots, t_k)$ and $t'_i \sim_{\mathcal{A}} t_i$ for $i \in \{1, \dots, k\}$. Observe that $t_i^{\mathcal{A}} = t_i'^{\mathcal{A}}$ since $\mathcal{A} \models t_i = t'_i$ for all $i \in \{i, \dots, k\}$. Hence,

$$\begin{aligned}
&\mathcal{A} \models P(t_1, \dots, t_k) \\
\text{iff } &\langle t_1^{\mathcal{A}}, \dots, t_k^{\mathcal{A}} \rangle \in P^{\mathcal{A}} \\
\text{iff } &\langle t_1'^{\mathcal{A}}, \dots, t_k'^{\mathcal{A}} \rangle \in P^{\mathcal{A}} \\
\text{iff } &\mathcal{A} \models P(t'_1, \dots, t'_k)
\end{aligned}$$

3. Let φ be a satisfiable closed τ formula in Skolem normal form in first-order logic with equality and let $\mathcal{A}$ be a model of φ. Let $\mathcal{U}_\tau$ all ground terms of τ (as usual we assume that τ contains at least one constant symbol). We construct a Herbrand structure $\mathcal{H}$ with universe $\mathcal{U}_{/\sim_\mathcal{A}} = \{[t]_{\sim_\mathcal{A}} : t \in \mathcal{U}\}$. First, we set for every constant symbol $c \in \tau$ that $c^\mathcal{H} = [c]_{\sim_\mathcal{A}}$. Moreover, let $f \in \tau$ be a function symbol with arity k. Then, we set $f^\mathcal{H}([t_1]_{\sim_\mathcal{A}}, \ldots, [t_k]_{\sim_\mathcal{A}}) = [f(t_1, \ldots, t_k)]_{\sim_\mathcal{A}}$. Finally, we fix for every predicate symbol $P \in \tau$ with arity ℓ that

$$\langle [t_1]_{\sim_\mathcal{A}}, \ldots, [t_\ell]_{\sim_\mathcal{A}} \rangle \in P^\mathcal{H} \text{ iff } \mathcal{A} \models P(t_1, \ldots, t_\ell).$$

Using question 2., we observe that this definition actually is well-defined; that is, it does not matter which representative of the equivalence classes we choose since they all behave the same w.r.t. interpretation of P under $\mathcal{A}$.

It remains to show that $\mathcal{H} \models \varphi$. The proof, however, is very similar to the proof from the lecture. Hence, we may simply adapt it. Therefore, we proceed by induction on the number of universal quantifications in φ. The base case postulates no quantification in φ which is then a Boolean combination of terms of the form $P(t_1, \ldots, t_k)$ or $t_1 = t_2$. However, by construction of $\mathcal{H}$ we see that $\mathcal{H} \models P(t_1, \ldots, t_k)$ if and only if $\mathcal{A} \models P(t_1, \ldots, t_k)$ and, more interestingly, $\mathcal{A} \models t_1 = t_2$ if and only if $t_1 \sim_\mathcal{A} t_2$ if and only if $[t_1]_{\sim_\mathcal{A}} = [t_2]_{\sim_\mathcal{A}}$ if and only if $\mathcal{H} \models t_1 = t_2$. Therefore, since $\mathcal{A} \models \varphi$ we get $\mathcal{H} \models \varphi$.

Assume $\mathcal{H}$ is a model for those closed formulae with n universal quantifications for which $\mathcal{A}$ is a model. We establish now that for every formula with $n+1$ universal quantifications for which $\mathcal{A}$ is a model $\mathcal{H}$ is a model too. Let now $\varphi = \forall x_0 \psi$ where $\psi = \forall x_1 \ldots \forall x_n \eta$ where η is a quantifier free formula. Assume $\mathcal{A} \models \varphi$. Pick an arbitrary term $t \in \mathcal{U}_\tau$ and consider the formula $\psi[t/x_0]$. i.e. we substituted every occurence of x_0 with t. Since $\mathcal{A} \models \varphi$ also $\mathcal{A} \models \psi[t/x_0]$. By induction hypothesis we get $\mathcal{H} \models \psi[t/x_0]$ since $\mathcal{A} \models \psi$ and ψ contains n universal quantifications. However, since $t^\mathcal{H} = [t]_{\sim_\mathcal{A}}$ we get get that $\mathcal{H}_{x_0 \mapsto [t]_{\sim_\mathcal{A}}} \models \psi$. By the arbitrary choice of t we observe that for every $t \in \mathcal{U}_\tau$ we have $\mathcal{H}_{x_0 \mapsto [t]_{\mathcal{A}\sim}} \models \psi$ and, consequently,

$$\mathcal{H} \models \underbrace{\forall x_0 \psi}_{=\varphi}.$$

Hence, we may state that for every closed sastisfiable formula in Skolem normal form in first-order logic with equality there is a Herbrand model of the form above.

4. Our example is $\varphi = \forall x \forall y ((x = y) \wedge (f(x) = x))$ which is satisfiable with $\mathcal{A} = \langle \{a\}, \{a \mapsto a\} \rangle$. We now add one arbitrary constant symbol c to $\mathcal{A}$ such that $c^\mathcal{A} = a$, i.e. $\mathcal{A} = \langle \{a\}, \{a \mapsto a\}, a \rangle$. The set of ground terms becomes $\mathcal{U} = \{c, f(c), f(f(c)), \ldots\}$. However, we observe that $\mathcal{A} \models t = f(t)$ for every ground term $t \in \mathcal{U}$. Consequently, we get $\sim_\mathcal{A} = \mathcal{U} \times \mathcal{U}$ by simple inductive reasoning. Hence, $\mathcal{H} = \langle \{[c]_{\sim_\mathcal{A}}\}, \{[c]_{\sim_\mathcal{A}} \mapsto [c]_{\sim_\mathcal{A}}\}, [c]_{\sim_\mathcal{A}} \rangle$. We observe that $\mathcal{H} \models \varphi$ and, moreover, $\mathcal{A}$ and $\mathcal{H}$ are isomorphic.

5. (a) Let $\mathcal{U}^{\mathcal{A}} = \mathcal{U}^{\mathcal{B}} = \mathbb{Z}$ We choose $f^{\mathcal{A}}\colon x \mapsto x$ as the identity function and $f^{\mathcal{B}}\colon x \mapsto -x$ as negation. Both are clearly self inverse.

   (b) Since F does not contain a constant, we add the constant c and extend our models with $c^{\mathcal{A}} c^{\mathcal{B}} = 1$. The set of ground terms generated by $\{f, c\}$ is:

$$\mathcal{U} = \{f^k(c) \mid k \in \mathbb{N}\}$$

   In $\mathcal{A}$ $f(x) = x$ therefore $[f^k(c)]_{\sim_{\mathcal{A}}} = [f^{k'}(c)]_{\mathcal{A}} = \mathcal{U}$ and hence

$$\mathcal{U}_{/\sim_{\mathcal{A}}} = \{\mathcal{U}\}$$

   In $\mathcal{B}$ we have $f^k(c) = f^{k'}(c)$ iff $k \equiv k' \pmod 2$ which results in $\mathcal{U}_{/\sim_{\mathcal{B}}} = \{[c]_{\sim_{\mathcal{B}}}, [f(c)]_{\sim_{\mathcal{B}}}\}$ with the following two equivalence classes:

$$[c]_{\sim_{\mathcal{B}}} = \{f^{2k}(c) \mid k \in \mathbb{N}\}$$
$$[f(c)]_{\sim_{\mathcal{B}}} = \{f^{2k+1}(c) \mid k \in \mathbb{N}\}$$