Exercise sheet: Resolution and Unification

Exercise 1: Exclusive Barbers' Club

There is a town with an exclusive Barbers' club. A barber is a member of the club if and only if he doesn't shave people who shave him (the barber). The barber Arturo claims, that he shaves all members of the club and no one else.

(a) Give an informal proof that Arturo is lying.

(b) Formalize the membership property and Arturo's claim.

(c) Use ground resolution to show that the conjunction of both formulae is unsatisfiable.

Solution

(a) Assume Arturo is telling the truth. Then, by the membership condition of the barbers' club, none of the members of the barber shop shave Arturo. It follows that Arturo shaves only people who do not shave him, so Arturo is himself a member of the club. However, now there exists a barber (Arturo) whom Arturo shaves (by his claim) and who is shaven by Arturo. Which is a contradiction to Arturo being a member of the club.

(a) Membership: $\forall x(M(x) \leftrightarrow \forall y(S(x,y) \rightarrow \neg S(y,x)))$
Claim: $\forall x(S(a,x) \leftrightarrow M(x))$

(a) First we convert both formulae to the required skolem form with matrix in CNF:

$$\begin{aligned}
&\forall x(M(x) \leftrightarrow \forall y(S(x,y) \rightarrow \neg S(y,x)))\\
\equiv&\forall x((M(x) \vee \neg\forall y(\neg S(x,y) \vee \neg S(y,x))) \wedge (\neg M(x) \vee \forall y(\neg S(x,y) \vee \neg S(y,x))))\\
\equiv&\forall x((M(x) \vee \neg\forall y(\neg S(x,y) \vee \neg S(y,x))) \wedge (\neg M(x) \vee \forall z(\neg S(x,z) \vee \neg S(z,x))))\\
\equiv&\forall x((M(x) \vee \exists y(S(x,y) \wedge S(y,x))) \wedge (\neg M(x) \vee \forall z(\neg S(x,z) \vee \neg S(z,x))))\\
\equiv&\forall x\exists y\forall z((M(x) \vee S(x,y) \wedge S(y,x)) \wedge (\neg M(x) \vee z\neg S(x,z) \vee \neg S(z,x)))\\
\rightsquigarrow&\forall x\forall z((M(x) \vee S(x,f(x)) \wedge S(f(x),x)) \wedge (\neg M(x) \vee z\neg S(x,z) \vee \neg S(z,x)))\\
\equiv&\forall x\forall z((M(x) \vee S(x,f(x))) \wedge (M(x) \vee S(f(x),x)) \wedge (\neg M(x) \vee z\neg S(x,z) \vee \neg S(z,x)))
\end{aligned}$$

$$\begin{aligned}
&\forall x(S(a,x) \leftrightarrow M(x))\\
\equiv&\forall x((\neg S(a,x) \vee M(x)) \wedge (S(a,x) \vee \neg M(x)))
\end{aligned}$$

This gives us the following five clauses :

$$\begin{aligned}
\mathcal{C}_1 &= \{M(x), S(x, f(x))\} \\
\mathcal{C}_2 &= \{M(x), S(f(x), x)\} \\
\mathcal{C}_3 &= \{\neg M(x), \neg S(x, z), \neg S(z, x)\} \\
\mathcal{C}_4 &= \{M(x), \neg S(a, x)\} \\
\mathcal{C}_5 &= \{\neg M(x), S(a, x)\}
\end{aligned}$$

We now have derive the empty clause:

$C_1 = \mathcal{C}_1[a/x] = \{M(a), S(a, f(a))\}$

$C_2 = \mathcal{C}_2[a/x] = \{M(a), S(f(a), a)\}$

$C_3 = \mathcal{C}_3[a/x, a/z] = \{\neg M(a), \neg S(a, a)\}$

$C_4 = \mathcal{C}_5[a/x] = \{\neg M(a), S(a, a)\}$

$C_5 = \{\neg M(a)\}$ — From C_3 and C_4

$C_6 = \mathcal{C}_3[f(a)/x, a/z] = \{\neg M((f(a)), \neg S(f(a), a), \neg S(a, f(a))\}$

$C_7 = \mathcal{C}_4[f(a)/x] = \{M(f(a)), \neg S(a, f(a))\}$

$C_8 = \{\neg S(f(a), a), \neg S(a, f(a))\}$ — From C_6 and C_7

$C_9 = \{\neg S(f(a), a), M(a)\}$ — From C_1 and C_8

$C_{10} = \{M(a)\}$ — From C_2 and C_9

$C_{11} = \square$ — From C_5 and C_{10}

Exercise 2: Herbrand's theorem

The Herbrand's theorem as we have seen it in Lecture 10, holds only for formulas of first-order theory without equality. Give a formula in first-order logic with equality for which Herbrand's theorem does not hold.

Solution

Pick $F = \forall x \forall y((x = y) \wedge (f(x) = x))$ for which $\mathcal{A}$ with $U_{\mathcal{A}} = \{a\}$ and $f^{\mathcal{A}}\colon a \mapsto a$ is a model.

The corresponding Herbrand structure is $\mathcal{H}$ with $U_{\mathcal{H}} = \{c, f(c), f(f(c)), f(f(f(c))), \ldots\}$ and $f^{\mathcal{H}}(t) = f(t)$. However, $F \models \psi$ where $\psi = \forall x \forall y(x = y)$. Since ψ is true if and only if the structure has exactly one element, $\mathcal{H} \not\models \psi$ and therefore $\mathcal{H} \not\models F$.

Exercise 3: Occurence check

During the unification algorithm it is checked whether a term contains the variable it is replacing. This is called the "occurence check". Assume now a unification algorithm which omits this occurence check. Then, give a set $\mathbb{L} = \{L_1, L_2\}$ such that L_1 and L_2 do not share a variable and cannot be unified but the (modified) unification algorithm gives either that $\mathbb{L}$ can be modified or runs into an infinite loop.

Solution

Consider

$$L_1 = P(y, f(y)) \text{ and } L_2 = P(x, f(f(x))).$$

The initial step of the unification algorithm identifies x and y in some fashion. W.l.o.g. we assume $[x/y]$ which then yields

$$L_1[x/y] = P(x, f(x)) \text{ and } L_2[x/y] = P(x, f(f(x))).$$

In the next step, the occurence of $\underline{x}$ in $P(x, f(\underline{x}))$ is unified with the term $\underline{f(x)}$ in $P(x, f(\underline{f(x)}))$. By the lack of an occurence check it is undetected that this substitution either yields an infinite sequence of further substitutions or that the resulting substitution is not correct.

Exercise 4: Happy Dragons

Note: This exercise requires the resolution algorithm for predicate logic, which you will only cover in the lecture on Tuesday, July 1st.

Express the following facts by formulas in predicate logic.

(a) Every dragon is happy if all its children can fly.

(b) Green dragons can fly.

(c) A dragon is green if it is a child of at least one green dragon.

Prove by resolution that the conjunction of these three statements implies the following: all green dragons are happy.

Solution

We use unary predicates H, G and F to describe that a dragon is happy, green, and it can fly, respectively, and a binary predicate C to describe a dragon being a child of another dragon. Then the sentences in English can be expressed as follows:

(a) $F_1 = \forall x \,\big(\forall y \,(C(y, x) \to F(y)) \to H(x)\big)$

(b) $F_2 = \forall x \,(G(x) \to F(x))$

(c) $F_3 = \forall x \,\big(\exists y \,(C(x, y) \land G(y)) \to G(x)\big)$

(d) $F_4 = \forall x \,(G(x) \to H(x))$

We need to prove that the last formula is entailed by the previous three, formally, $F_1 \land F_2 \land F_3 \vdash F_4$. Equivalently, we prove that $F_1 \land F_2 \land F_3 \land \neg F_4$ is unsatisfiable.

First we transform each formula into the required Skolem form with matrices in CNF:

$$F_1 \equiv \forall x \exists y \, \big((C(y,x) \vee H(x)) \wedge (\neg F(y) \vee H(x)) \big)$$
$$\equiv_s \forall x \, \big((C(f(x),x) \vee H(x)) \wedge (\neg F(f(x)) \vee H(x)) \big)$$

$$F_2 \equiv \forall x \, (\neg G(x) \vee F(x))$$

$$F_3 \equiv \forall x \forall y \, \big(\neg C(x,y) \vee \neg G(y) \vee G(x) \big)$$

$$\neg F_4 \equiv \exists x \, \big(G(x) \wedge \neg H(x) \big)$$
$$\equiv_s G(a) \wedge \neg H(a)$$

Finally, we prove the required by presenting a resolution proof of the empty clause: