Exercise sheet: First order logic

Exercise 1: Defining properties

1. Define that the unary function f is injective using a formula in first-order logic with equality.
2. Define that the unary function f is surjective using a formula in first-order logic with equality.
3. Give a satisfiable formula in first-order logic with equality which is only satisfiable by structures with infinite universes.
4. Give a satisfiable forumula in first-order logic **without equality** which is only satisfiable by structures with infinite universes.

Exercise 2: Modeling

The teachers of one kindergarten came up with a strategy to improve the discipline within the group of kids. They promised a price for those who behave well. Here is what they announced:

1. All kids who do their homework will receive a cake.
2. Every kid that does not start a fight against any other kid will receive a cake.
3. There is (at least one) kid who does the homework and against whom no other kid starts a fight.

One kid concluded the following: every kid that received a cake, did not start a fight with any other kid.

Prove that this conclusion is wrong. Give three formulas in first-order logic, T_1, T_2, T_3 that represent the statements given by the teachers, and one formula K that represents the conclusion by the kid. Use predicate symbols H, F, C for doing homework, starting a fight with someone, and receiving a cake, respectively. Show that

$$T_1, T_2, T_3 \not\models K$$

by giving a structure $\mathcal{A}$ with a finite universe $U_{\mathcal{A}}$ such that $\mathcal{A} \models T_i$ for $1 \leq i \leq 3$ but $\mathcal{A} \not\models K$.

Exercise 3: Semantics of first-order logic

Among the following 5 formulas, observe all 10 pairs. For every two formulas decide if they are equivalent or not. If they are equivalent, prove this using the known transformations from the lecture. If they are not equivalent, give a structure that is a model for one of them but not for the other.

$$F_1 = \forall x \exists y (P(x) \wedge Q(y))$$
$$F_2 = (\forall x P(x) \wedge \exists y Q(y))$$
$$F_3 = (\forall x Q(y) \wedge \exists y P(x))$$
$$F_4 = \forall x (Q(y) \wedge \exists y P(x))$$
$$F_5 = \exists y \forall x (P(x) \wedge Q(y))$$

Exercise 4: Normal forms

Let Q be a ternary, R, E binary and P, S unary predicates. Translate the following formulas to rectified form, then to prenex form, and finally to Skolem form:

1. $$\forall z\, \exists y\, (Q(x, g(y), z) \vee \neg \forall x\, P(x)) \wedge \neg \forall z\, \exists x\, \neg R(f(x, z), z)$$

2. $$\forall y \neg ((R(b, g(x)) \vee \forall x P(f(x))) \wedge S(y))\,.$$

3. $$\forall x (P(x) \rightarrow \exists x \forall y (E(y, x) \rightarrow P(x))) \vee \neg \forall x \forall y (\neg P(x) \vee x = y \vee \neg P(y))$$