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EXERCISE SHEET: FIRST ORDER LOGIC

Exercise 1: Defining properties

1. Define that the unary function f is injective using a formula in first-order logic with equality.
2. Define that the unary function f is surjective using a formula in first-order logic with equality.
3. Give a satisfiable formula in first-order logic with equality which is only satisfiable by structures with infinite universes.
4. Give a satisfiable forumula in first-order logic **without equality** which is only satisfiable by structures with infinite universes.

Solution

1. $\varphi_i = \forall x \forall y\, (f(x) = f(y) \to x = y)$, i.e. every pair of values which are mapping to the same element coincide.
2. $\varphi_s = \forall x \exists y f(y) = x$, i.e. for every element y exists one element x which is mapped to y under f.
3. $\varphi_i \land \neg\varphi_s$ which expresses that the function f is injective but not surjective. Essential to the argument is now that any injective function $f\colon A \to A$ for a finite A necessarily is surjective because every element $a \in A$ needs to be mapped to a unique element $f(a) \in A$. If we now assume that f is not surjective then there is $b \in A$ which is not mapped to from an element $a \in A$. By the pidgeon hole principle there is no injective mapping from A to $A \setminus \{b\}$.

   However, this formula is satisfiable. Consider for this the structure $\mathcal{A} = \langle \mathbb{N}, f \rangle$ with $f_{\mathcal{A}}\colon \mathbb{N} \to \mathbb{N}$ such that $f_{\mathcal{A}}(n) = n + 1$. This $f_{\mathcal{A}}$ is obviously injective but nothing maps to 0 rendering $f_{\mathcal{A}}$ not surjective.
4.

$$\begin{aligned} \psi = {} & \forall x P(x, f(x)) && (1) \\ & \land\ \forall x \neg P(f(x), x) && (2) \\ & \land\ \forall x \forall y \forall z (P(x,y) \land P(y,z) \to P(x,z)) && (3) \\ & \land\ \exists x \top && (4) \end{aligned}$$

   Assume ψ had a model $\mathcal{A} = (U, I)$ for finite but not empty U. Then there exists $x \in U$ and a number $k \in \mathbb{N}^+$ such that $(f^{\mathcal{A}})^k(x) = x$ (i.e. the graph

$\big(U, \{(x, f^{\mathcal{A}}(x)) \mid x \in U\}\big)$ contains a cycle). In order to satisfy clauses (1) and (3), it must hold that $P^{\mathcal{A}}(x, (f^{\mathcal{A}})^{k-1}) = 1$, however, this also means that $P^{\mathcal{A}}((f^{\mathcal{A}})^{k}, (f^{\mathcal{A}})^{k-1}) = 1$, which violates clause (2). If U is empty, then clause (4) cannot be satisfied. One possible model for ψ is $\langle \mathbb{Z}, f, P\rangle$ where $f^{\mathcal{A}}(x) = x + 1$ and $P^{\mathcal{A}} = \{(x, y) \in \mathbb{Z}^2 \mid x < y\}$.

Exercise 2: Modeling

The teachers of one kindergarten came up with a strategy to improve the discipline within the group of kids. They promised a price for those who behave well. Here is what they announced:

1. All kids who do their homework will receive a cake.
2. Every kid that does not start a fight against any other kid will receive a cake.
3. There is (at least one) kid who does the homework and against whom no other kid starts a fight.

One kid concluded the following: every kid that received a cake, did not start a fight with any other kid.

Prove that this conclusion is wrong. Give three formulas in first-order logic, T_1, T_2, T_3 that represent the statements given by the teachers, and one formula K that represents the conclusion by the kid. Use predicate symbols H, F, C for doing homework, starting a fight with someone, and receiving a cake, respectively. Show that

$$T_1, T_2, T_3 \not\models K$$

by giving a structure $\mathcal{A}$ with a finite universe $U_{\mathcal{A}}$ such that $\mathcal{A} \models T_i$ for $1 \le i \le 3$ but $\mathcal{A} \not\models K$.

Solution

The statements a modeled as follows:

$$\begin{aligned} T_1 &= \forall x(H(x) \to C(x)) \\ T_2 &= \forall x(\forall y \neg F(x, y) \to C(x)) \\ T_3 &= \exists x(H(x) \wedge \forall y \neg F(y, x)) \\ K &= \forall x(C(x) \to \neg\exists y F(x, y)) \end{aligned}$$

We can take a structure with two kids, $U_{\mathcal{A}} = \{Alice, Bob\}$, with $H^{\mathcal{A}} = \{\text{Bob}\}$, $C^{\mathcal{A}} = \{\text{Alice}, \text{Bob}\}$, and $F^{\mathcal{A}} = \{(\text{Bob}, \text{Alice})\}$. In this case we have $\mathcal{A}(T_1) = \mathcal{A}(T_2) = \mathcal{A}(T_3) = 1$ and $\mathcal{A}(K) = 0$.

Exercise 3: Semantics of first-order logic

Among the following 5 formulas, observe all 10 pairs. For every two formulas decide if they are equivalent or not. If they are equivalent, prove this using the known transformations from the lecture. If they are not equivalent, give a structure that is a model for one of them but not for the other.

$$\begin{aligned}
F_1 &= \forall x \exists y (P(x) \wedge Q(y)) \\
F_2 &= (\forall x P(x) \wedge \exists y Q(y)) \\
F_3 &= (\forall x Q(y) \wedge \exists y P(x)) \\
F_4 &= \forall x (Q(y) \wedge \exists y P(x)) \\
F_5 &= \exists y \forall x (P(x) \wedge Q(y))
\end{aligned}$$

Solution

F_1, F_2 and F_5 are equivalent[1]:

$$\begin{aligned}
F_1 &= \forall x \exists y (P(x) \wedge Q(y)) & \\
&= \forall x (P(x) \wedge \exists y Q(y)) & (y \text{ is not free in } P(x)) \\
&= (\forall x P(x) \wedge \exists y Q(y)) & (x \text{ is not free in } \exists y Q(y)) \\
&= F_2 & \\
&= \exists y (\forall x P(x) \wedge Q(y)) & (y \text{ is not free in } \forall x P(x)) \\
&= \exists y \forall x (P(x) \wedge Q(y)) & (x \text{ is not free in } Q(y)) \\
&= F_5 &
\end{aligned}$$

Formula F_3 is not equivalent to F_1, F_2, nor F_5, since the structure $\mathcal{A}$ with $U_{\mathcal{A}} = \{a, b\}$, $P^{\mathcal{A}} = \{a, b\}$, $Q^{\mathcal{A}} = \{b\}$, $x^{\mathcal{A}} = a$, $y^{\mathcal{A}} = a$, is a model for F_1 (and thus also for F_2 and F_5), but not for F_3.

The same structure is not a model for F_4, so F_4 is not equivalent to F_1, F_2, nor F_5.

Formulas F_3 and F_4 are not equivalent, since the structure $\mathcal{A}$ with $U_{\mathcal{A}} = \{a, b\}$, $P^{\mathcal{A}} = \{a\}$, $Q^{\mathcal{A}} = \{b\}$, $x^{\mathcal{A}} = a$, $y^{\mathcal{A}} = b$, is a model for F_3, but not for F_4.

Exercise 4: Normal forms

Let Q be a ternary, R, E binary and P, S unary predicates. Translate the following formulas to rectified form, then to prenex form, and finally to Skolem form:

1. $$\forall z\, \exists y\, (Q(x, g(y), z) \vee \neg \forall x\, P(x)) \wedge \neg \forall z\, \exists x\, \neg R(f(x, z), z)$$

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[1] IMPORTANT: Although F_1 and F_5 differ only in the order of quantifiers and they are equivalent, this is not a rule. In general, changing the order of quantifiers **does not** yield an equivalent formula. For example, $\forall x \exists y R(x, y) \not\equiv \exists y \forall x R(x, y)$. You can prove this by giving a structure that is a model for one, but not for the other formula (for example, natural numbers and relation $<$).

2. $$\forall y \neg ((R(b, g(x)) \vee \forall x P(f(x))) \wedge S(y))\,.$$

3. $$\forall x(P(x) \to \exists x \forall y(E(y,x) \to P(x))) \vee \neg \forall x \forall y(\neg P(x) \vee x = y \vee \neg P(y))$$

Solution

1. Recall that a formula is rectified if no variable appears both bound and free and if each bound variable has exactly one binding occurrence. An equivalent rectified form of the given formula is obtained by renaming some of the bound variables as follows:

   $$\forall z\, \exists y\, (Q(x, g(y), z) \vee \neg \forall u\, P(u)) \wedge \neg \forall w\, \exists v\, \neg R(f(v,w), w)\,.$$

   To obtain an equivalent prenex form, first push the negations inward:

   $$\forall z\, \exists y\, (Q(x, g(y), z) \vee \exists u\, \neg P(u)) \wedge \exists w\, \forall v\, R(f(v,w), w)\,.$$

   Then bring the quantifiers to the front (using the equivalences from Lecture 10):

   $$\forall z\, \exists y\, \exists u\, \exists w\, \forall v\, ((Q(x, g(y), z) \vee \neg P(u)) \wedge R(f(v,w), w))\,.$$

   Finally we obtain an equisatisfiable Skolem form with new unary function symbols f_1, f_2, f_3:

   $$\forall z\, \forall v\, ((Q(x, g(f_1(z)), z) \vee \neg P(f_2(z))) \wedge R(f(v, f_3(z)), f_3(z)))\,.$$

2. $$\begin{aligned} &\forall y \neg ((R(b, g(x)) \vee \forall x P(f(x))) \wedge S(y)) \\ \equiv &\forall y \neg ((R(b, g(x)) \vee \forall z P(f(z))) \wedge S(y)) && \text{(Renaming)} \end{aligned}$$

   This gives us a rectified formula. We continue towards a prenex form.

   $$\begin{aligned} &\forall y \neg \big((R(b, g(x)) \vee \forall z P(f(z))) \wedge S(y)\big) \\ \equiv &\forall y \big((\neg R(b, g(x)) \wedge \neg \forall z P(f(z))) \vee \neg S(y)\big) && \text{(DeMorgan laws)} \\ \equiv &\forall y \big((\neg R(b, g(x)) \wedge \exists z \neg P(f(z))) \vee \neg S(y)\big) && \text{(Negat. and quant.)} \\ \equiv &\forall y \exists z \big((\neg R(b, g(x)) \wedge \neg P(f(z))) \vee \neg S(y)\big) && \text{(Quant. to the front)} \end{aligned}$$

   This formula is in a prenex normal form. We continue towards a Skolem form.

   $$\begin{aligned} &\forall y \exists z \big((\neg R(b, g(x)) \wedge \neg P(f(z))) \vee \neg S(y)\big) \\ \equiv &\forall y \big((\neg R(b, g(x)) \wedge \neg P(f(h(y)))) \vee \neg S(y)\big) && \text{(Skolemization)} \end{aligned}$$

3.

$$\begin{aligned}
&\forall x(P(x) \to \exists x \forall y(E(y,x) \to P(x))) \lor \neg \forall x \forall y(\neg P(x) \lor x = y \lor \neg P(y)) && \\
\equiv&\forall x(\neg P(x) \lor \exists x \forall y(\neg E(y,x) \lor P(x))) \lor \neg \forall x \forall y(\neg P(x) \lor x = y \lor \neg P(y)) && \text{translated } \to \\
\equiv&\forall x(\neg P(x) \lor \exists x \forall y(\neg E(y,x) \lor P(x))) \lor \exists x \exists y(P(x) \land x \neq y \land P(y)) && \text{moved } \neg \text{ to atoms} \\
\equiv&\forall x_1(\neg P(x_1) \lor \exists x_2 \forall x_3(\neg E(x_3,x_2) \lor P(x_2))) \lor \exists x_4 \exists x_5(P(x_4) \land x_4 \neq x_5 \land P(x_5)) && \text{rectified} \\
\equiv&\forall x_1 \exists x_2 \forall x_3 \exists x_4 \exists x_5((\neg P(x_1) \lor \neg E(x_3,x_2) \lor P(x_2)) \lor (P(x_4) \land x_4 \neq x_5 \land P(x_5))) && \text{prenexed} \\
\equiv&\forall x_1 \forall x_3((\neg P(x_1) \lor \neg E(x_3, f_2(x_1)) \lor P(f_2(x_1)))) && \\
&\lor (P(f_4(x_1,x_3)) \land f_4(x_1,x_3) \neq f_5(x_1,x_3) \land P(f_5(x_1,x_3))) && \text{skolemized}
\end{aligned}$$