Exercise sheet: Propositional Proof Theory

Exercise 1: Currying

Prove that for any $n \geq 1$, the following formula has a sequent calculus proof:

$$(A_1 \land (A_2 \land (\cdots \land A_n) \cdots) \to B) \to (A_1 \to A_2 \to \cdots \to A_n \to B)$$

Remember that $A \to B \to C = A \to (B \to C)$

Solution

We first prove the following:

Lemma 1. *for all* Γ, Δ *it holds:*

$$A_1, A_2, \ldots A_n, \Gamma \vdash_G \bigwedge_{i=1}^{n} A_i, \Delta$$

Proof. Proof by induction on n:

- Case $n = 1$

$$\frac{}{A_1, \Gamma \implies A_1, \Delta}\ \mathsf{Ax}$$

- Case $n = m + 1$

$$\frac{\dfrac{}{A_1, \ldots, A_m, A_{m+1}, \Gamma \implies A_{m+1}, \Delta}\ \mathsf{Ax} \qquad \dfrac{}{A_1, \ldots, A_m, A_{m+1}, \Gamma \implies \bigwedge_{i=1}^{m} A_i, \Delta}\ \mathsf{IH}}{A_1, \ldots, A_m, A_{m+1}, \Gamma \implies \bigwedge_{i=1}^{m+1} A_i, \Delta}\ \land\mathsf{R}$$

□

To prove the final claim, note that by n-fold application of the rule $\to \mathsf{R}$ it suffices to show $A_1, \ldots, A_n, \bigwedge_{i=1}^{n} \to B \vdash_G B$, which easily follows from Lemma 1:

$$\frac{\dfrac{}{A_1, \ldots, A_n \vdash_G \bigwedge_{i=1}^{n} A_i, B}\ \text{Lemma 1} \qquad \dfrac{}{B, A_1, \ldots, A_n \implies B}\ \mathsf{Ax}}{A_1, \ldots, A_n, \bigwedge_{i=1}^{n} \to B \vdash_G B}\ \to \mathsf{L}$$

Exercise 2: Natural Deduction can Simulate Sequent Calculus

Give a constructive proof that if $\Gamma \vdash_G \Delta$ then $\Gamma \vdash_N \bigvee \Delta$.

Solutiom

We first prove the following useful rules:

$$\dfrac{F \lor G \qquad \neg F}{G}\ \lor\neg\mathsf{E}_1 \qquad\qquad \dfrac{F \lor G \qquad \neg G}{F}\ \lor\neg\mathsf{E}_2 \qquad\qquad \dfrac{}{F \lor \neg F}\ \mathsf{LEM}$$

We start with the proofs for $\lor\neg\mathsf{E}_1$ and $\lor\neg\mathsf{E}_2$:

$$\dfrac{F \lor G \qquad \dfrac{\dfrac{\neg F \qquad [F]_1}{\bot}\ \neg\mathsf{E}}{G}\ \bot(2) \qquad [G]_1}{G}\ \lor\mathsf{E}(1) \qquad\qquad \dfrac{F \lor G \qquad [F]_1 \qquad \dfrac{\dfrac{\neg G \qquad [G]_1}{\bot}\ \neg\mathsf{E}}{F}\ \bot(2)}{F}\ \lor\mathsf{E}(1)$$

And now the proof for LEM:

$$\dfrac{\dfrac{\dfrac{\dfrac{[\neg(F \lor \neg F)]_1 \qquad \dfrac{[\neg F]_2}{F \lor \neg F}\ \lor\mathsf{I}_2}{\bot}\ \neg\mathsf{E}}{F}\ \bot(2) \qquad \dfrac{\dfrac{[\neg(F \lor \neg F)]_1 \qquad \dfrac{[F]_3}{F \lor \neg F}\ \lor\mathsf{I}_1}{\bot}\ \neg\mathsf{E}}{\neg F}\ \neg\mathsf{I}(3)}{\bot}\ \neg\mathsf{E}}{F \lor \neg F}\ \bot(1)$$

Now we can prove the existance of a natural deduction proof tree by structural induction on the sequent calculus proof tree.

Case 1: $\bot\mathsf{L}$

$$\dfrac{\bot}{\bigvee\Delta}\ \bot(1)$$

Case 2: Ax

$$\dfrac{}{A, \Gamma \implies A, \Delta}\ \mathsf{Ax} \qquad \rightsquigarrow \qquad \dfrac{A}{A \lor \bigvee\Delta}\ \lor\mathsf{I}_1$$

Case 3: $\dfrac{\Gamma \vdash_G F, \Delta}{\neg F, \Gamma \implies \Delta}\ \neg\mathsf{L}$

By induction hypothesis we have $T\colon \Gamma \vdash_N F \lor \bigvee\Delta$

$$\dfrac{T \qquad \dfrac{[F]_1 \qquad \neg F}{\bot}\ \neg\mathsf{E} \qquad [\bigvee\Delta]_1}{\bigvee\Delta}\ \lor\mathsf{E}(1)$$

Case 4: $\dfrac{F, \Gamma \vdash_G \Delta}{\Gamma \implies \neg F, \Delta}\ \neg\mathsf{R}$

By induction hypothesis we have $T : F, \Gamma \vdash_N \bigvee\Delta$

$$\dfrac{\dfrac{}{F \lor \neg F}\ \mathsf{LEM} \qquad \dfrac{[T]_{F:1}}{\neg F \lor \bigvee\Delta}\ \lor\mathsf{I}_1 \qquad \dfrac{[\neg F]_1}{\neg F \lor \bigvee\Delta}\ \lor\mathsf{I}_2}{\neg F \lor \bigvee\Delta}\ \lor\mathsf{E}(1)$$

Case 5: $\dfrac{F, G, \Gamma \vdash_G \Delta}{F \land G, \Gamma \implies \Delta}\ \land\mathsf{L}$

By induction hypothesis we have a proof tree $T\colon F, G, \Gamma \vdash_n \bigvee\Delta$. By replacing every open occurance of F in T with the prooftree $\dfrac{F \land G}{F}\ \land\mathsf{E}_1$, and every open occurance of G with $\dfrac{F \land G}{G}\ \land\mathsf{E}_2$ we obtain $T'\colon F \land G, \Gamma \vdash_N \Delta$.

Case 6: $\dfrac{\Gamma \vdash_G F, \Delta \qquad \Gamma \vdash_G G, \Delta}{\Gamma \implies F \wedge G, \Delta} \wedge\mathsf{R}$

By induction hypothesis we have $T_1 \colon \Gamma \vdash_N F \vee \bigvee \Delta$ and $T_2 \colon \Gamma \vdash_N G \vee \bigvee \Delta$

$$\dfrac{\dfrac{}{\bigvee \Delta \vee \neg \bigvee \Delta}\,\mathsf{LEM} \qquad \dfrac{[\bigvee \Delta]_1}{F \wedge G \vee \bigvee \Delta}\,\vee\mathsf{I}_2 \qquad \dfrac{\dfrac{\dfrac{\mathsf{T}_1 \qquad [\neg \bigvee \Delta]_1}{F}\,\vee\neg\mathsf{E}_2 \qquad \dfrac{\mathsf{T}_2 \qquad [\neg \bigvee \Delta]_1}{G}\,\vee\neg\mathsf{E}_2}{F \wedge G}\,\wedge\mathsf{I}}{F \wedge G \vee \bigvee \Delta}\,\vee\mathsf{I}_1}{F \wedge G \vee \bigvee \Delta}\,\vee\mathsf{E}(1)$$

Case 7: $\dfrac{F, \Gamma \vdash_G \Delta \qquad G, \Gamma \vdash_G \Delta}{F \vee G, \Gamma \implies \Delta} \vee\mathsf{L}$

By induction hypothesis we have $T_1 \colon F, \Gamma \vdash_G \bigvee \Delta$ and $T_2 \colon G, \Gamma \vdash_G \bigvee \Delta$

$$\dfrac{F \vee G \qquad [T_1]_{F:1} \qquad [T_2]_{G:1}}{\bigvee \Delta}\,\vee\mathsf{E}(1)$$

Case 8: $\dfrac{\Gamma \vdash_G F, G, \Delta}{\Gamma \implies F \vee G, \Delta} \vee\mathsf{R}$

By induction hypothesis we have $T \colon \Gamma \vdash_N F \vee G \vee \bigvee \Delta$. This is already the proof tree we need.

Case 9: $\dfrac{\Gamma \vdash_G F, \Delta \qquad G, \Gamma \vdash_G \Delta}{(F \to G), \Gamma \implies \Delta} \to \mathsf{L}$

By induction hypothesis we have $T_1 \colon \Gamma \vdash_N F \vee \bigvee \Delta$ and $T_2 \colon G, \Gamma \vdash_N \bigvee \Delta$ We obtain $T_2' \colon F, (F \to G), \Gamma \vdash_N \bigvee \Delta$ from T_2 by replacing every open occurance of G with the proof tree $\dfrac{F \to G \qquad F}{G} \to \mathsf{E}$. Then we construct

$$\dfrac{T_1 \qquad [T_2']_{F:1} \qquad [\bigvee \Delta]_1}{\bigvee \Delta}\,\vee\mathsf{E}(1)$$

Case 10: $\dfrac{F, \Gamma \vdash_G G, \Delta}{\Gamma \implies (F \to G), \Delta} \to \mathsf{R}$

By induction hypothesis we have $T : F, \Gamma \vdash_N G \vee \bigvee \Delta$.

$$\dfrac{\dfrac{}{\bigvee \Delta \vee \neg \bigvee \Delta}\,\mathsf{LEM} \qquad \dfrac{[\bigvee \Delta]_1}{(F \to G) \vee \bigvee \Delta}\,\vee\mathsf{I}_2 \qquad \dfrac{\dfrac{\dfrac{[T]_{F:2} \qquad [\neg \bigvee \Delta]_1}{G}\,\vee\neg\mathsf{E}_2}{F \to G}\to\mathsf{I}(2)}{(F \to G) \vee \bigvee \Delta}\,\vee\mathsf{I}_1}{(F \to G) \vee \bigvee \Delta}\,\vee\mathsf{E}(1)$$

Exercise 3: Hilbert System

Solution

Prove that $\vdash_H F \wedge G \to G \wedge F$. You may use the deduction theorem. By the deduction theorem it suffices to show that $F \wedge G \vdash_H G \wedge F$

(0)	$F \wedge G$	By assumption
(1)	F	from A_4 and (0)
(2)	G	from A_5 and (0)
(3)	$F \to G \wedge F$	from A_3 and (2)
(4)	$G \wedge F$	from (3) and (1)