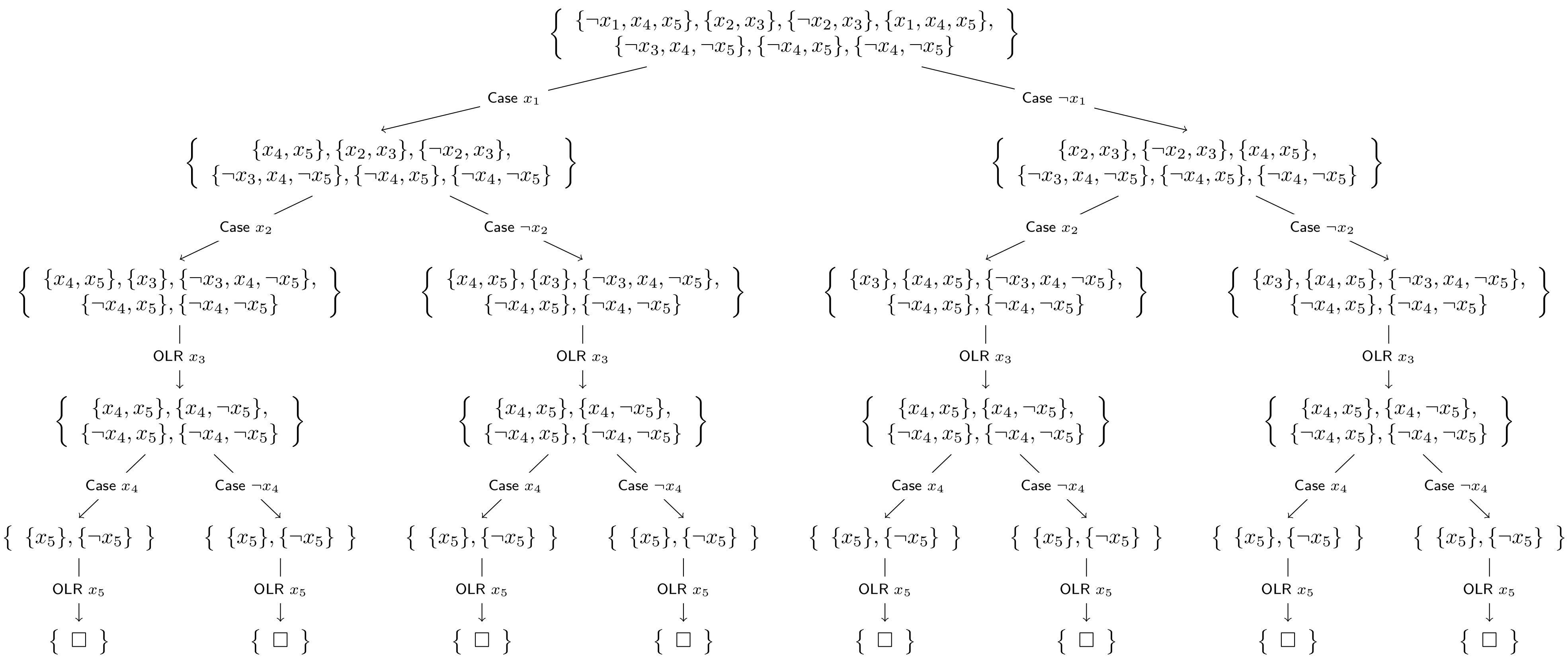
Exercise sheet: Resolution and DPLL

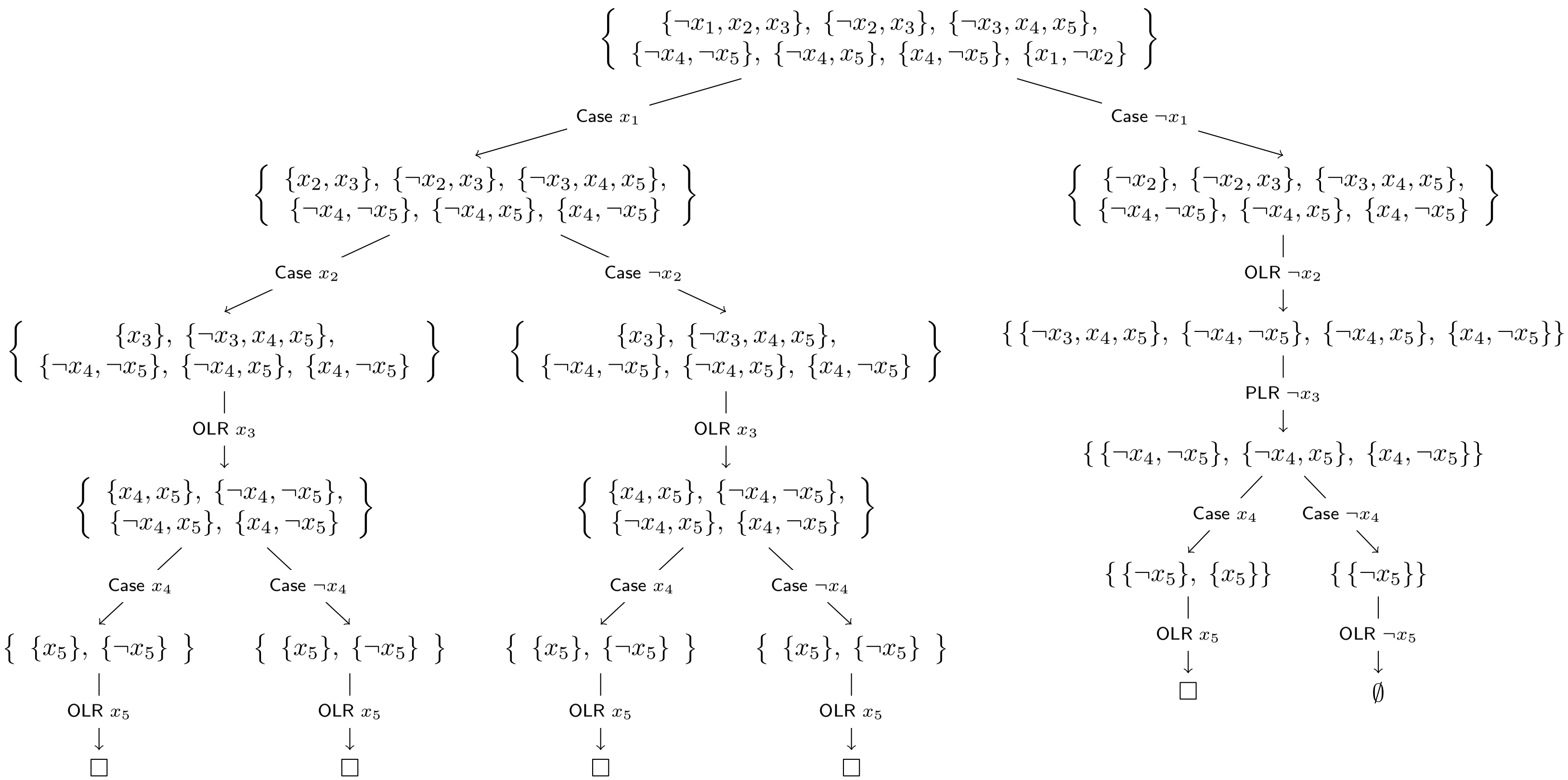
Exercise 1: DPLL

1. Apply the DPLL algorithm to the following sets of clauses. Use OLR first, then PLR. Whenever it is possible to choose multiple literals, choose the variable with the smallest index first, and choose x_i before $\neg x_i$.
   a) $\{\{\neg x_1, x_4, x_5\}, \{x_2, x_3\}, \{\neg x_2, x_3\}, \{x_1, x_4, x_5\}, \{\neg x_3, x_4, \neg x_5\}, \{\neg x_4, x_5\}, \{\neg x_4, \neg x_5\}\}$
   b) $\{\{\neg x_1, x_2, x_3\}, \{\neg x_2, x_3\}, \{\neg x_3, x_4, x_5\}, \{\neg x_4, \neg x_5\}, \{\neg x_4, x_5\}, \{x_4, \neg x_5\}, \{x_1, \neg x_2\}\}$
2. Apply the CDCL algorithm using resolution for clause learning to the same formulae.

1.

Solution


{ {¬x1, x4, x5}, {x2, x3}, {¬x2, x3}, {x1, x4, x5}, {¬x3, x4, ¬x5}, {¬x4, x5}, {¬x4, ¬x5} }
Case x1
Case ¬x1
{ {x4, x5}, {x2, x3}, {¬x2, x3}, {¬x3, x4, ¬x5}, {¬x4, x5}, {¬x4, ¬x5} }
{ {x2, x3}, {¬x2, x3}, {x4, x5}, {¬x3, x4, ¬x5}, {¬x4, x5}, {¬x4, ¬x5} }
Case x2
Case ¬x2
{ {x4, x5}, {x3}, {¬x3, x4, ¬x5}, {¬x4, x5}, {¬x4, ¬x5} }
{ {x3}, {x4, x5}, {¬x3, x4, ¬x5}, {¬x4, x5}, {¬x4, ¬x5} }
OLR x3
{ {x4, x5}, {x4, ¬x5}, {¬x4, x5}, {¬x4, ¬x5} }
Case x4
Case ¬x4
{ {x5}, {¬x5} }
OLR x5
{ □ }


{¬x1, x2, x3}, {¬x2, x3}, {¬x3, x4, x5}, {¬x4, ¬x5}, {¬x4, x5}, {x4, ¬x5}, {x1, ¬x2}
Case x1
Case ¬x1
{x2, x3}, {¬x2, x3}, {¬x3, x4, x5}, {¬x4, ¬x5}, {¬x4, x5}, {x4, ¬x5}
{¬x2}, {¬x2, x3}, {¬x3, x4, x5}, {¬x4, ¬x5}, {¬x4, x5}, {x4, ¬x5}
Case x2
Case ¬x2
OLR ¬x2
{x3}, {¬x3, x4, x5}, {¬x4, ¬x5}, {¬x4, x5}, {x4, ¬x5}
{{¬x3, x4, x5}, {¬x4, ¬x5}, {¬x4, x5}, {x4, ¬x5}}
OLR x3
PLR ¬x3
{x4, x5}, {¬x4, ¬x5}, {¬x4, x5}, {x4, ¬x5}
{{¬x4, ¬x5}, {¬x4, x5}, {x4, ¬x5}}
Case x4
Case ¬x4
{x5}, {¬x5}
{{¬x5}, {x5}}
{{¬x5}}
OLR x5
OLR ¬x5

2. a)

0 –	$\{\neg x_1, x_4, x_5\}$	$\{x_2, x_3\}$	$\{\neg x_2, x_3\}$	$\{x_1, x_4, x_5\}$	$\{\neg x_3, x_4, \neg x_5\}$	$\{\neg x_4, x_5\}$	$\{\neg x_4, \neg x_5\}$
1 Case x_1	$\{x_4, x_5\}$	$\{x_2, x_3\}$	$\{\neg x_2, x_3\}$		$\{\neg x_3, x_4, \neg x_5\}$	$\{\neg x_4, x_5\}$	$\{\neg x_4, \neg x_5\}$
2 Case x_2	$\{x_4, x_5\}$		$\{x_3\}$		$\{\neg x_3, x_4, \neg x_5\}$	$\{\neg x_4, x_5\}$	$\{\neg x_4, \neg x_5\}$
3 OLR x_3	$\{x_4, x_5\}$				$\{x_4, \neg x_5\}$	$\{\neg x_4, x_5\}$	$\{\neg x_4, \neg x_5\}$
4 Case x_4						$\{x_5\}$	$\{\neg x_5\}$
5 OLR x_5							$\square$

We learn the following new clause:

$\{\neg x_4, \neg x_5\}$	Resolve with $\{\neg x_4, x_5\}$
$\{\neg x_4\}$	Cannot resolve with $\{\neg x_2, x_3\}$

and backtrack to step 1:

0 –	$\{\neg x_1, x_4, x_5\}$	$\{x_2, x_3\}$	$\{\neg x_2, x_3\}$	$\{x_1, x_4, x_5\}$	$\{\neg x_3, x_4, \neg x_5\}$	$\{\neg x_4, x_5\}$	$\{\neg x_4, \neg x_5\}$	$\{\neg x_4\}$
1 OLR $\neg x_4$	$\{\neg x_1, x_5\}$	$\{x_2, x_3\}$	$\{\neg x_2, x_3\}$	$\{x_1, x_5\}$	$\{\neg x_3, \neg x_5\}$			
2 Case x_1	$\{x_5\}$	$\{x_2, x_3\}$	$\{\neg x_2, x_3\}$		$\{\neg x_3, \neg x_5\}$			
3 OLR x_5		$\{x_2, x_3\}$	$\{\neg x_2, x_3\}$		$\{\neg x_3\}$			
4 OLR $\neg x_3$		$\{x_2\}$	$\{\neg x_2\}$					
5 OLR x_2			$\square$					

We learn the following new clause:

$\{\neg x_2, x_3\}$	Resolve with $\{x_2, x_3\}$
$\{\neg x_3\}$	Resolve with $\{\neg x_3, x_4, \neg x_5\}$
$\{x_4, \neg x_5\}$	Resolve with $\{\neg x_1, x_4, x_5\}$
$\{\neg x_1, x_4\}$	Resolve with $\{\neg x_4\}$
$\{\neg x_1\}$	

and backtrack to step 2:

1 OLR $\neg x_4$	$\{\neg x_1, x_5\}$	$\{x_2, x_3\}$	$\{\neg x_2, x_3\}$	$\{x_1, x_5\}$	$\{\neg x_3, \neg x_5\}$	$\{\neg x_1\}$
1 OLR $\neg x_1$		$\{x_2, x_3\}$	$\{\neg x_2, x_3\}$	$\{x_5\}$	$\{\neg x_3, \neg x_5\}$	
2 OLR x_5		$\{x_2, x_3\}$	$\{\neg x_2, x_3\}$		$\{\neg x_3\}$	
3 OLR $\neg x_3$		$\{x_2\}$	$\{\neg x_2\}$			
5 OLR x_2			$\square$			

b)

0	$\{\neg x_1, x_2, x_3\}$	$\{\neg x_2, x_3\}$	$\{\neg x_3, x_4, x_5\}$	$\{\neg x_4, \neg x_5\}$	$\{\neg x_4, x_5\}$	$\{x_4, \neg x_5\}$	$\{x_1, \neg x_2\}$
1 Case x_1	$\{x_2, x_3\}$	$\{\neg x_2, x_3\}$	$\{\neg x_3, x_4, x_5\}$	$\{\neg x_4, \neg x_5\}$	$\{\neg x_4, x_5\}$	$\{x_4, \neg x_5\}$	
1 Case x_2		$\{x_3\}$	$\{\neg x_3, x_4, x_5\}$	$\{\neg x_4, \neg x_5\}$	$\{\neg x_4, x_5\}$	$\{x_4, \neg x_5\}$	
1 OLR x_3			$\{x_4, x_5\}$	$\{\neg x_4, \neg x_5\}$	$\{\neg x_4, x_5\}$	$\{x_4, \neg x_5\}$	
1 Case x_4				$\{\neg x_5\}$	$\{x_5\}$		
1 OLR x_5				$\square$			

We learn the following new clause:

$\{\neg x_4, \neg x_5\}$	Resolve with $\{\neg x_4, x_5\}$
$\{\neg x_4\}$	Cannot resolve with $\{\neg x_2, x_3, \neg x_5\}$

and backtrack to step 1:

0	$\{$	$\{\neg x_1, x_2, x_3\}$	$\{\neg x_2, x_3\}$	$\{\neg x_3, x_4, x_5\}$	$\{\neg x_4, \neg x_5\}$	$\{\neg x_4, x_5\}$	$\{x_4, \neg x_5\}$	$\{x_1, \neg x_2\}$	$\{\neg x_4\}$	$\}$
1 OLR $\neg x_4$	$\{$	$\{\neg x_1, x_2, x_3\}$	$\{\neg x_2, x_3\}$	$\{\neg x_3, x_5\}$			$\{\neg x_5\}$	$\{x_1, \neg x_2\}$		$\}$
2 OLR $\neg x_5$	$\{$	$\{\neg x_1, x_2, x_3\}$	$\{\neg x_2, x_3\}$	$\{\neg x_3\}$				$\{x_1, \neg x_2\}$		$\}$
2 OLR $\neg x_3$	$\{$	$\{\neg x_1, x_2\}$	$\{\neg x_2\}$					$\{x_1, \neg x_2\}$		$\}$
2 OLR $\neg x_2$	$\{$	$\{\neg x_1\}$								$\}$
2 OLR $\neg x_1$	$\{$									$\}$

Exercise 2: Input resolution

Let $\mathbb{C} = \{C_1, \ldots, C_n\}$ be a set of clauses. We say a sequence $\langle B_0, B_1, \ldots, B_k\rangle$ is an *input resolution* of B_k from $\mathbb{C}$ if the following holds:

(i) $B_0 \in \mathbb{C}$, and

(ii) $B_{\ell+1}$ is the resolvent of B_ℓ and one clause from $\mathbb{C}$.

1. Show that there is a resolution of $\square$ but no input resolution of $\square$ from

$$F = \{\{\neg z, x\}, \{\neg x, \neg y\}, \{y, z\}, \{z, \neg y, x\}, \{y, \neg x\}\}.$$

2. Show that there is an input resolution of $\square$ from

$$\{\{\neg t, \neg y\}, \{\neg y, z\}, \{\neg x, \neg z, t\}, \{\neg x, y\}, \{x\}\}.$$

3. Prove that input resolution is complete for Horn formulae by showing that there is an input resolution of $\square$ from any unsatisfiable set of clauses where every clause contains at most one positive literal.[1]

Hint

Consider a Horn formula as a conjunction of *rules* $\bigwedge A \to H$ with A being a set of propositional variables and H being either a propositional variable or 0, i.e. the *procedural reading* of Horn formulae. Try to consider an "inverse" marking algorithm which starts from 0 and marks propositional variables on the left hand side of implications until a proof why 0 is implied by any possible assignment which sets the right hand sides of rules $1 \to H \equiv \bigwedge \emptyset \to H$ to true is found. Then, frame the executed steps in terms of input resolution.

Solution

1. Since for input resolution one of the clauses is part of the input and every clause in the input contains at least two distinct literals all resolvents contain at least one literal.

2.

[1] Note that input resolution is not complete in general because F from 1. is unsatisfiable but there is no input resolution proof of that fact.

$$\dfrac{\dfrac{\dfrac{\dfrac{\{\neg t, \neg y\} \qquad \{\neg x, \neg z, t\}}{\{\neg x, \neg z, \neg y\}} \qquad \{\neg y, z\}}{\{\neg x, \neg y\}} \qquad \{\neg x, y\}}{\{\neg x\}} \qquad \{x\}}{\square}$$

3. Let F be an unsatisfiable Horn formula. We view F as a set of clauses of the form

$$\bigwedge H \to y$$

where H is a set of variables and y is either a variable or $\bot$. Let $\mathsf{Mb}(F, M)$ be the result of running the model building algorithm on F starting with the initial marking M (Usually $M = \emptyset$, but we will need to argue about M inductively). By the completeness of the model building algorithm we obtain a sequence of markings $M_0, M_1, \ldots, M_k$ and a sequence of clauses $C_0, C_1, \ldots C_k$ with $C_i = \bigwedge H_i \to y_i$ such that the following properties hold:

(i) $M_0 = M$

(ii) $H_i \subseteq M_i$

(iii) $M_{i+1} = M_i \cup \{y_i\}$ for $i < k$

(iv) $y_k = \bot$

For any set of variables A, we define $\overline{A} \coloneqq \{\neg x \mid x \in A\}$. Now we prove by induction on k the existance of an input resolution from F to $\overline{I}$ for some $I \subseteq M$

Case 0**:** then by Properties (ii) and (iv) $\overline{M_0} = C_0 \in F$

Case $k+1$**:** Since $\mathsf{Mb}(F, M_0) = \mathsf{Mb}(F, M_1) = \mathsf{UNSAT}$ it follows by induction hypothesis that there exists an input resolution from F to $\overline{I_1}$ with $I_1 \subseteq M_1$. If $y_0 \in I_1$ then we can resolve $\overline{I_1}$ with C_0 to obtain $\overline{I_0} \coloneqq \overline{I_1 \backslash \{y_0\}} \cup \overline{H_0}$ (equivalently $I_0 = I_1 \backslash \{y_0\} \cup H_0$). Due to properties (ii), (iii) and $I_1 \subseteq M_1$ it follows that $I_0 \subseteq M_0$. If $y_0 \notin I_1$ then we already have $I_1 \subseteq M_0$.