Exercise sheet: Compactness theorem for propositional logic

Exercise 1: Compactness

1. Suppose that $\mathcal{S} \models F$ for some formula F and set of formulas $\mathcal{S}$. Show that there is a finite set $\mathcal{S}_0 \subseteq \mathcal{S}$ such that $\mathcal{S}_0 \models F$.

2. Given an undirected graph $G = (V, E)$, a set of vertices $S \subseteq V$ is a *clique* if every pair of distinct vertices $u, v \in S$ are connected by an edge and S is an *independent set* if no pair of distinct vertices $u, v \in S$ is connected by an edge. Now consider the following two statements:

   (A) Every infinite graph either has an infinite clique or an infinite independent set.

   (B) For all k there exists n such that any graph with n vertices has a clique of size k or an independent set of size k.

   The goal of this question is to show that (A) implies (B).

   a) Carefully formulate the negation of (B).

   b) Assuming the negation of (B), use the Compactness Theorem to prove the negation of (A), i.e., that there is an infinite graph with no infinite clique and no infinite independent set.

Exercise 2: Graph Homomorphism

Let $H = \langle V_H, E_H \rangle$ be a finite graph. We say a graph $G = \langle V_G, E_G \rangle$ is homomorphic to H if there exists a function $f\colon V_G \to V_H$ such that for every $\langle u, v \rangle \in E_G$ already $\langle f(u), f(v) \rangle \in E_H$.

1. Show that every graph G is homomorphic to $U = \langle \{0\}, \{\langle 0, 0 \rangle\} \rangle$.

2. Use the compactness theorem for propositional logic to prove that a graph G is homomorphic to H if and only if every finite subgraph of G is homomorphic to H.

Exercise 3: Parity, Flip-Sets and Smurfs

Gargamel captured n Smurfs. Due to Gargamel's obsession with puzzles he forces the Smurfs to participate in a game. The Smurfs are arranged in a long line facing all in the same direction. Hence, the Smurf at the beginning of the line can see the back of all other Smurfs. In turn the Smurf at the end of the line can see no other Smurf.

Garagamel now puts hats on all the Smurfs. However, every Smurf cannot see their own hat but only the hats of the Smurfs in front of them. Gargamel reveals to the Smurfs that there are only two kinds of hats: blue and red ones. The game is now played in n rounds. In the i-th round the i-th Smurf (starting at the beginning of the line) is asked to guess the colour of their hat. If they guess correctly they will be let go. However, if they guess incorrectly they will be eaten by Azrael. All other Smurfs hear every taken guess.

1. [1] Design a strategy which allows to save at least $n-1$ Smurfs from Azrael. We assume that the Smurfs are informed of the rules of the game beforehand and can form a plan together accordingly.
2. We call a set $F \subseteq \{0,1\}^\omega$ a *flip-set* if for every pair $\alpha, \beta \in \{0,1\}^\omega$ such that α and β differ at exactly one position either $\alpha \in F$ or $\beta \in F$ but not both.

   Use the compactness theorem for propositional logic to prove the existence of a flip-set.

   Hint.

   You may use that the compactness theorem for propositional logic holds true for uncountable sets of formulae. For example a formula set which utilizes an uncountable set of propositional variables

   $$\mathbb{X} = \{X_\alpha : \alpha \in \{0,1\}^\omega\}$$

   where $\{0,1\}^\omega$ is the set of all infinite sequences of bits.
3. Design a strategy which would save at least all but one Smurf from being eaten by Azrael if Gargamel captured and forced a countable amount of Smurfs to participate in his game.

[1] Questions 1. and 3. are brain teaser rather than traditional exercises that we would ask in an exam.