Exercise sheet: Compactness theorem for propositional logic

Exercise 1: Compactness

1. Suppose that $\mathcal{S} \models F$ for some formula F and set of formulas $\mathcal{S}$. Show that there is a finite set $\mathcal{S}_0 \subseteq \mathcal{S}$ such that $\mathcal{S}_0 \models F$.

2. Given an undirected graph $G = (V, E)$, a set of vertices $S \subseteq V$ is a *clique* if every pair of distinct vertices $u, v \in S$ are connected by an edge and S is an *independent set* if no pair of distinct vertices $u, v \in S$ is connected by an edge. Now consider the following two statements:

   (A) Every infinite graph either has an infinite clique or an infinite independent set.

   (B) For all k there exists n such that any graph with n vertices has a clique of size k or an independent set of size k.

   The goal of this question is to show that (A) implies (B).

   a) Carefully formulate the negation of (B).

   b) Assuming the negation of (B), use the Compactness Theorem to prove the negation of (A), i.e., that there is an infinite graph with no infinite clique and no infinite independent set.

Solution

1. Note that $\mathcal{S} \models F$ iff $\mathcal{S} \cup \{\neg F\}$ is unsatisfiable. By the Compactness Theorem this holds iff $\mathcal{S}_0 \cup \{\neg F\}$ is unsatisfiable for some finite subset $\mathcal{S}_0$ of $\mathcal{S}$. But this is equivalent to $\mathcal{S}_0 \models F$.

2. a) The negation of (B) states that there exists k such that for all n there is a graph with n vertices that has no clique of size k nor independent set of size k.

   b) Assume that there exists k such that for all n there is a graph with n vertices that has no clique of size k nor independent set of size k. We construct an infinite graph with set of vertices $\mathbb{N}$ that has no clique nor independent set of size k.

   For each pair of natural numbers $i < j$ introduce a propositional variable $P_{i,j}$ (indicating whether or not there is an edge between i and j). Given a finite set $S \subseteq \mathbb{N}$ define

$$F_S := \bigvee_{\substack{i,j \in S \\ i<j}} P_{i,j} \wedge \bigvee_{\substack{i,j \in S \\ i<j}} \neg P_{i,j}$$

expressing that S is neither a clique nor independent set.

We first prove that there is an infinite graph with no clique of size k nor independent set of size k. The same graph does not have an infinite clique nor an infinite independent set.

The set of formulas $\Phi = \{F_S : S \subseteq \mathbb{N}, |S| = k\}$ is finitely satisfiable (every finite subset is satisfiable) by the assumption that the negation of (B) holds. Thus, by the Compactness Theorem, the set of formulas Φ must be satisfiable. Then a satisfying assignment yields the desired graph with no clique of size k and no independent set of size k, and therefore it also has no infinite clique and no infinite independent set.

Exercise 2: Graph Homomorphism

Let $H = \langle V_H, E_H \rangle$ be a finite graph. We say a graph $G = \langle V_G, E_G \rangle$ is homomorphic to H if there exists a function $f \colon V_G \to V_H$ such that for every $\langle u, v \rangle \in E_G$ already $\langle f(u), f(v) \rangle \in E_H$.

1. Show that every graph G is homomorphic to $U = \langle \{0\}, \{\langle 0, 0 \rangle\} \rangle$.

2. Use the compactness theorem for propositional logic to prove that a graph G is homomorphic to H if and only if every finite subgraph of G is homomorphic to H.

Solution

1. Let $G = \langle V, E \rangle$ be a graph. We choose the homomorphism $f \colon V \to \{0\}$ with $f(v) = 0$ for all $v \in V$. This is indeed a homomorphism since for every $\langle u, v \rangle \in E$ we have $\langle f(u), f(v) \rangle = \langle 0, 0 \rangle \in \{\langle 0, 0 \rangle\}$.

2. Let G and H be as in the exercise. But w.l.o.g. we assume $V_H \cap V_G = \emptyset$ and consequently $E_H \cap E_G = \emptyset$. We introduce a set of propositional variables

$$\mathbb{X} = \{T_{u,v} \colon \langle u, v \rangle \in V_G \times V_H\}.$$

We introduce

$$\begin{aligned}\Phi = &\{\bigvee_{v \in V_H} T_{u,v} \colon u \in V_G\} \cup \{\bigwedge_{\langle v_1, v_2 \rangle \in V_H \times V_H} \neg(T_{u,v_1} \wedge T_{u,v_2}) \colon u \in V_G\} \cup \\ &\{\bigvee_{\langle v_1, v_2 \rangle \in E_H} (T_{u_1,v_1} \wedge T_{u_2,v_2}) \colon \langle u_1, u_2 \rangle \in E_G\},\end{aligned}$$

which is due to the finitness of H well defined. It is essential to observe that any homomorphism $f \colon V_G \to V_H$ induces an interpretation $\mathcal{I}_f \colon \{T_{u,v} \colon \langle u, v \rangle \in V_G \times V_H\} \to \{0, 1\}$ with

$$I_f(T_{u,v}) = 1 \text{ iff } f(u) = v. \tag{1}$$

Moreover, any interpretation $\mathcal{I}_f\colon \{T_{u,v}\colon \langle u,v\rangle \in V_G \times V_H\} \to \{0,1\}$ induces a homomorphism f by using (1) if for every $u \in V_G$

$$|\{v \in V_H \mid \mathcal{I}_f(T_{u,v}) = 1\}| = 1. \tag{2}$$

However, every interpretation $\mathcal{I}_f$ with

$$\mathcal{I}_f \models \{ \bigvee_{v \in V_H} T_{u,v} \wedge \bigwedge_{\langle v_1,v_2\rangle \in V_H \times V_H} \neg(T_{u,v_1} \wedge T_{u,v_2})\colon u \in V_G\}$$

satisfies condition (2).

Using the compactness theorem for propositional logic yields the claim almost immediately. Every finite subset $\Phi_0 \subseteq \Phi$ only refers to finitely many propositional variables $\mathbb{X}_0$, hence there is a finite set $V_0 \subseteq V_G$ such that $\mathbb{X}_0 \subseteq \{T_{u,v}\colon \langle u,v\rangle \in V_0 \times V_H\}$. The subgraph of G which is induced by V_0 is homomorphic by a homomorphism f to H by assumption. Equation (1) induces an interpretation $\mathcal{I}\colon \mathbb{X}_0 \to \{0,1\}$ which satisfies Φ_0. Therefore, Φ is satisfiable as well.

On the other hand, if G is homomorphic to H then there is a homorphism f from G to H which renders every finite subgraph G_0 of G also homomorphic to H by f restricted to vertices from G_0.

Exercise 3: Parity, Flip-Sets and Smurfs

Gargamel captured n Smurfs. Due to Gargamel's obsession with puzzles he forces the Smurfs to participate in a game. The Smurfs are arranged in a long line facing all in the same direction. Hence, the Smurf at the beginning of the line can see the back of all other Smurfs. In turn the Smurf at the end of the line can see no other Smurf. Garagamel now puts hats on all the Smurfs. However, every Smurf cannot see their own hat but only the hats of the Smurfs in front of them. Gargamel reveals to the Smurfs that there are only two kinds of hats: blue and red ones. The game is now played in n rounds. In the i-th round the i-th Smurf (starting at the beginning of the line) is asked to guess the colour of their hat. If they guess correctly they will be let go. However, if they guess incorrectly they will be eaten by Azrael. All other Smurfs hear every taken guess.

1. [1] Design a strategy which allows to save at least $n-1$ Smurfs from Azrael. We assume that the Smurfs are informed of the rules of the game beforehand and can form a plan together accordingly.

2. We call a set $F \subseteq \{0,1\}^\omega$ a *flip-set* if for every pair $\alpha, \beta \in \{0,1\}^\omega$ such that α and β differ at exactly one position either $\alpha \in F$ or $\beta \in F$ but not both.

   Use the compactness theorem for propositional logic to prove the existence of a flip-set.

[1] Questions 1. and 3. are brain teaser rather than traditional exercises that we would ask in an exam.

Hint.

You may use that the compactness theorem for propositional logic holds true for uncountable sets of formulae. For example a formula set which utilizes an uncountable set of propositional variables

$$\mathbb{X} = \{X_\alpha \colon \alpha \in \{0,1\}^\omega\}$$

where $\{0,1\}^\omega$ is the set of all infinite sequences of bits.

3. Design a strategy which would save at least all but one Smurf from being eaten by Azrael if Gargamel captured and forced a countable amount of Smurfs to participate in his game.

Solution

1. Let $\overrightarrow{b_1 \dots b_n}$ with $b_i \in \{0,1\}$ for $1 \leq i \leq n$ be the representation of the n Smurfs where 0s represent *blue* hats while 1s represent *red* hats. The arrow indicates the direction in which the Smurfs are facing. In the first round Gargamel asks Smurf b_1 to state its guess. Since b_1 is aware of the values $b_2 \dots b_n$ its guesses 1 if and only if $|\{i \in \{2,\dots,n\} \mid b_i = 1\}|$ is odd. Lets call this guess g_1 (and from now on every subsequent guess g_i if made by Smurf i). Hence, the number of 1s in $g_1 b_2 \dots b_n$ is even. Moreover, at the i-th round the i-th Smurf is aware that the number of 1s in $g_1 \dots g_{i-1} b_i \dots b_n$ is even. Hence, the i-th Smurf (being aware of $g_1 \dots g_{i-1}$ and $b_{i+1} \dots b_n$) may deduce its hat colour and hence walk free.

   This strategy allows to save at least all Smurfs $b_2, \dots, b_n$.

2. We introduce a set of propositional variables

$$\mathbb{X} = \{X_\alpha \colon \alpha \in \{0,1\}^\omega\}\,.$$

   Now, we define

$$\Phi = \left\{ X_\alpha \leftrightarrow \neg X_\beta \colon \begin{array}{l} \langle \alpha, \beta \rangle \in \{0,1\}^\omega \times \{0,1\}^\omega \\ \text{such that } \alpha \text{ and } \beta \text{ differ at exactly one index.} \end{array} \right\}$$

   It is imminent that an interpretation $\mathcal{I} \colon \mathbb{X} \to \{0,1\}$ with $\mathcal{I} \models \Phi$ induces a flip-set $F = \{\alpha \in \{0,1\}^\omega \mid \mathcal{I}(X_\alpha) = 1\}$. It remains to show the satisfiability of Φ. To apply the compactness theorem we fix an arbitrary but finite subset $\Phi_0 \subseteq \Phi$. Let $\mathbb{X}_0 \subseteq \mathbb{X}$ be the finite set which contains all propositional variables occuring in Φ_0. Consider the graph

$$G = \left\langle \mathbb{X}_0, \left\{ \langle X_\alpha, X_\beta \rangle \colon \begin{array}{l} \langle X_\alpha, X_\beta \rangle \in \mathbb{X}_0 \times \mathbb{X}_0 \\ \text{such that } \alpha \text{ and } \beta \text{ differ at exactly one index.} \end{array} \right\} \right\rangle$$

   It is essential to observe that G is undirected and contains only circles of even length (since every bit change has to be reversed to get back to the original word).

Hence, the graph is 2-colourable which in turn induces a model for Φ_0. Therefore, Φ_0 is satisfiable and by arbitrary choice of Φ_0 every finite subset of Φ is. By compactness we get the satisfiability of Φ and therefore the existence of a flip-set.

3. Using a flip-set the strategy of (1) generalises to the countable infinite case. For this the Smurfs beforehand agree on a flip-set F. The first Smurf chooses its guess such that $g_1 b_2 b_3 \cdots \in F$. And from that point onwards the i-th Smurf can deduce the colour of its hat in the i-th round using the knowledge of $g_1 \dots g_{i-1}$ and $b_{i+1} b_{i+2} \dots$.