Exercise sheet: Propositional Logic

Exercise 1: CNF Conversion

(a) Prove that converting a formula into Negation Normal Form (NNF) terminates, by giving a weight function $w\colon \mathsf{formula} \to \mathbb{N}$ such that the following inequalities hold for all F, G:

$$w(\neg\neg F) > w(F)$$
$$w(\neg(F \vee G)) > w(\neg F \wedge \neg G)$$
$$w(\neg(F \wedge G)) > w(\neg F \vee \neg G)$$

$w(F)$ should be defined recursively using only addition, subraction, multiplication, division and exponentiation.

(b) Prove that the result of converting a formula into NNF is unique: Let $F \rightsquigarrow G$ denote that G is obtained from F by using double negation elimination (DNE), or one of De Morgan's laws. Prove that $\rightsquigarrow$ has the Church-Rosser property: if $F \rightsquigarrow G_1$ and $F \rightsquigarrow G_2$ then there exists a formula H such that $G_1 \rightsquigarrow^* H$ and $G_2 \rightsquigarrow^* H$.

(c) Prove that the second step of converting a Formula to CNF terminates, by giving a weight function $w\colon \mathsf{formula} \to \mathbb{N}$ such that the following inequalities hold for all F, G, H:

$$w(F \vee (G \wedge H)) > w((F \wedge G) \vee (F \wedge H))$$
$$w((F \wedge G) \vee H) > w((F \wedge H) \vee (G \wedge H))$$

$w(F)$ should be defined recursively using only addition, subraction, multiplication, division and exponentiation.

(d) **Challenge:** find a weight function that fulfills the requirements of both (a) and (c).
Note: We do not have a solution for this.

Exercise 2: Large disjunctive normal form

1. Write down a **DNF**-formula equivalent to $(a_1 \vee b_1) \wedge (a_2 \vee b_2) \wedge \cdots \wedge (a_n \vee b_n)$.
2. Prove that any **DNF**-formula equivalent to the above formula must have at least 2^n clauses.

Exercise 3: Perfect matching

A **perfect matching** in an undirected graph $G = (V, E)$ is a subset of the edges $M \subseteq E$ such that every vertex $v \in V$ is an endpoint of exactly one edge in M. Given a finite graph G, describe how to obtain a propositional formula F_G such that F_G is satisfiable if and only if G has a perfect matching. The formula F_G should be computable from G in time polynomial in $|V|$.