Exercise sheet: Propositional Logic

Exercise 1: Validity and Satisfiability

Which of the following formulae are valid, and which are satisfiable? Give a short proof of your claim.

(a) $F_1 :\equiv ((p \to q) \to p) \to p$

(b) $F_2 :\equiv ((p \leftrightarrow q) \to r) \to (p \leftrightarrow (q \to r))$

(c) $F_3 :\equiv (p \leftrightarrow (q \to r)) \to ((p \leftrightarrow q) \to r)$

(d) $F_4 :\equiv (\neg p \vee q) \leftrightarrow (p \wedge \neg q)$

Exercise 2: Facts and deductions

Let F, G and H be formulas and let $\mathcal{S}$ be a set of formulas. Which of the following statements are true? Justify your answer.

(a) If F is unsatisfiable, then $\neg F$ is valid.

(b) If $F \to G$ is satisfiable and F is satisfiable, then G is satisfiable.

(c) $\mathcal{S} \models F$ and $\mathcal{S} \models \neg F$ cannot both hold.

(d) If $\mathcal{S} \models F \vee G$, $\mathcal{S} \cup \{F\} \models H$ and $\mathcal{S} \cup \{G\} \models H$, then $\mathcal{S} \models H$.

(e) Assume $F, G \models H$, $F, H \models G$, and $H, G \models F$. Then F, G, H are all equivalent.

Exercise 3: Equivalences

Prove that $\{\mathsf{nand}\}$ is a basis for propositional logic, i.e for every formula F there is an equivalent formula F' using only the nand operator. You may use the fact that $\{\wedge, \neg\}$ is a basis.

Exercise 4: Counting Models

Let F be a formula where every operator is $\leftrightarrow$.

(a) Prove that $\leftrightarrow$ is commutative (i.e $F \leftrightarrow G \equiv G \leftrightarrow F$ for all formulae F, G) and associative (i.e. $F \leftrightarrow (G \leftrightarrow H) \equiv (F \leftrightarrow G) \leftrightarrow H$).

(b) Prove that F is either valid or has an equivalent formula in the following normal form: Let $x_0, x_1, x_2, \dots$ be an enumeration of all variables. A formula φ is in normal form, if either $\varphi = x_i$ for a variable x_i, or $\varphi = x_i \leftrightarrow \psi$ where x_i is a variable and ψ is a formula in normal form, where for all $x_j \in \mathrm{Vars}(\psi)$ it holds that $j < i$.

(c) Prove that either F is valid, or exactly half of all assignments satisfy F.