Exercise sheet: Propositional Logic

Exercise 1: Validity and Satisfiability

Which of the following formulae are valid, and which are satisfiable? Give a short proof of your claim.

(a) $F_1 \coloneqq ((p \rightarrow q) \rightarrow p) \rightarrow p$

(b) $F_2 \coloneqq ((p \leftrightarrow q) \rightarrow r) \rightarrow (p \leftrightarrow (q \rightarrow r))$

(c) $F_3 \coloneqq (p \leftrightarrow (q \rightarrow r)) \rightarrow ((p \leftrightarrow q) \rightarrow r)$

(d) $F_4 \coloneqq (\neg p \vee q) \leftrightarrow (p \wedge \neg q)$

Solution

(a) Valid. Any Assignment $\mathcal{A}$ for which $\mathcal{A}(F_1) = 0$ would have to set $\mathcal{A}(p) = 0$ and $\mathcal{A}((p \rightarrow q) \rightarrow p) = 1$. This in turn is only possible if $\mathcal{A}(p \rightarrow q) = 0$, which is impossible when $\mathcal{A}(p) = 1$.

(b) Satisfiable but not valid. F_2 is satisfied by $\mathcal{A}(x) = 1$ for $x \in \{p, q, r\}$, but not by $\mathcal{A}(x) = 0$ for $x \in \{p, q, r\}$.

(c) Valid. From $\mathcal{A}(F_3) = 0$ it follows that $\mathcal{A}(p \leftrightarrow (q \rightarrow r)) = 1$ and $\mathcal{A}((p \leftrightarrow q) \rightarrow r) = 0$. This in turn implies that $\mathcal{A}(r) = 0$ and $\mathcal{A}(p \leftrightarrow q) = 1$, i.e. $\mathcal{A}(p) = \mathcal{A}(q)$. However from $\mathcal{A}(p \leftrightarrow (q \rightarrow r)) = 1$ it follows that either $\mathcal{A}(p) = 1$ and $\mathcal{A}(q \rightarrow r) = 1$ or $\mathcal{A}(p) = 0$ and $\mathcal{A}(q \rightarrow r) = 0$. The first case is impossible as we already deduced $\mathcal{A}(q) = \mathcal{A}(p) = 1$ and $\mathcal{A}(r) = 0$. The second case is also impossible as from $\mathcal{A}(q) = \mathcal{A}(p) = 0$ it already follows that $\mathcal{A}(q \rightarrow r) = 1$.

(d) Unsatisfiable. Proof by truth table:

p	q	$((\neg$	(p)	$\vee$	$q)$	$\leftrightarrow$	$(p$	$\wedge$	$\neg$	$(q)))$
0	0	1	0	1	0	0	0	0	1	0
0	1	1	0	1	1	0	0	0	0	1
1	0	0	1	0	0	0	1	1	1	0
1	1	0	1	1	1	0	1	0	0	1

Exercise 2: Facts and deductions

Let F, G and H be formulas and let $\mathcal{S}$ be a set of formulas. Which of the following statements are true? Justify your answer.

(a) If F is unsatisfiable, then $\neg F$ is valid.

(b) If $F \rightarrow G$ is satisfiable and F is satisfiable, then G is satisfiable.

(c) $\mathcal{S} \models F$ and $\mathcal{S} \models \neg F$ cannot both hold.

(d) If $\mathcal{S} \models F \vee G$, $\mathcal{S} \cup \{F\} \models H$ and $\mathcal{S} \cup \{G\} \models H$, then $\mathcal{S} \models H$.

(e) Assume $F, G \models H$, $F, H \models G$, and $H, G \models F$. Then F, G, H are all equivalent.

Solution

(a) True. Let $\mathcal{A}$ be an arbitrary assignment. Since F is unsatisfiable we have $\mathcal{A}(F) = 0$ and thus $\mathcal{A}(\neg F) = 1$.

(b) False. A counterexample is $P \rightarrow \bot$ for an atomic proposition P.

(c) False. If $\mathcal{S}$ is unsatisfiable then $\mathcal{S} \models F$ and $\mathcal{S} \models \neg F$ for any F.

(d) True. Let $\mathcal{A}$ be a model of $\mathcal{S}$. Since $\mathcal{S} \models F \vee G$, $\mathcal{A}$ is a model of F or a model of G. In the first case, since $\mathcal{S} \cup \{F\} \models H$, $\mathcal{A}$ is a model of H. Likewise in the second case, since $\mathcal{S} \cup \{G\} \models H$, $\mathcal{A}$ is a model of H. Since the two cases are exhaustive, $\mathcal{A}$ is a model of H. Thus every model of $\mathcal{S}$ is a model of H.

(e) False. A counterexample is $F = G = \bot$ and $H = \top$

Exercise 3: Equivalences

Prove that $\{\mathsf{nand}\}$ is a basis for propositional logic, i.e for every formula F there is an equivalent formula F' using only the nand operator. You may use the fact that $\{\wedge, \neg\}$ is a basis.

Solution

Proof by structural induction on F

Case $F = x$ Then F is already in the desired form.

Case $F = \neg G$ By IH there exists a formula $G' \equiv G$ using only the nand operator. Then $F \equiv G' \ \mathsf{nand}\ G'$

Case $F = G \wedge H$ By IH there exist formulae $G' \equiv G$ and $H' \equiv H$ using only the nand operator. Then $F \equiv \neg(G \mathsf{nand} H) \equiv \neg(G' \mathsf{nand} H') \equiv (G' \mathsf{nand} H') \mathsf{nand} (G' \mathsf{nand} H')$

Exercise 4: Counting Models

Let $F \neq \bot$ be a formula where every operator is $\leftrightarrow$.

(a) Prove that $\leftrightarrow$ is commutative (i.e $F \leftrightarrow G \equiv G \leftrightarrow F$ for all formulae F, G) and associative (i.e. $F \leftrightarrow (G \leftrightarrow H) \equiv (F \leftrightarrow G) \leftrightarrow H$).

(b) Prove that F is either valid or has an equivalent formula in the following normal form: Let $x_0, x_1, x_2, \ldots$ be an enumeration of all variables. A formula φ is in normal form, if either $\varphi = x_i$ for a variable x_i, or $\varphi = x_i \leftrightarrow \psi$ where x_i is a variable and ψ is a formula in normal form, where for all $x_j \in \mathrm{Vars}(\psi)$ is holds that $j < i$.

(c) Prove that either F is valid, or exactly half of all assignments satisfy F.

Solution

(a) $\leftrightarrow$ is commutative as for any formulae F, G and any assignment $\mathcal{A}$ $\mathcal{A}(F \leftrightarrow G) = \mathcal{A}(G \leftrightarrow F)$ by definition.

$\leftrightarrow$ is associative:

$$\begin{array}{ll}
 & \mathcal{A}(F \leftrightarrow (G \leftrightarrow H)) = 1 \\
\textsf{iff} & \mathcal{A}(F) = \mathcal{A}(G \leftrightarrow H) \\
\textsf{iff} & \mathcal{A}(F) = 1 \text{ and } \mathcal{A}(G) = \mathcal{A}(H) \text{ or } \mathcal{A}(F) = 0 \text{ and } \mathcal{A}(G) \neq \mathcal{A}(H) \\
\textsf{iff} & \mathcal{A}(F) = 1 \text{ and } \mathcal{A}(G) = 0 \text{ and } \mathcal{A}(H) = 0 \\
 & \text{or } \mathcal{A}(F) = 1 \text{ and } \mathcal{A}(G) = 1 \text{ and } \mathcal{A}(H) = 1 \\
 & \text{or } \mathcal{A}(F) = 0 \text{ and } \mathcal{A}(G) = 0 \text{ and } \mathcal{A}(H) = 1 \\
 & \text{or } \mathcal{A}(F) = 0 \text{ and } \mathcal{A}(G) = 1 \text{ and } \mathcal{A}(H) = 0 \\
\textsf{iff} & \mathcal{A}(F) = \mathcal{A}(G) \text{ and } \mathcal{A}(H) = 1 \text{ or } \mathcal{A}(F) \neq \mathcal{A}(G) \text{ and } \mathcal{A}(H) = 0 \\
\textsf{iff} & \mathcal{A}(F \leftrightarrow G) = \mathcal{A}(H) \\
\textsf{iff} & \mathcal{A}((F \leftrightarrow G) \leftrightarrow H) = 1
\end{array}$$

Alternatively: compare truth tables

(b) Proof by strong induction on $\max\{i \in \mathbb{N} \mid x_i \in \mathrm{Vars}(F)\}$

Case 0 Only the variable x_0 occurs. Then F is either valid, or $F \equiv x_0$. Proof by structural induction:

Case $F = x_0$ Then trivially $F \equiv x_0$.

Case $F = G \leftrightarrow H$ By induction hypothesis G and H are both either valid or equivalent to x_0. If both are valid or both are equivalent to x_0 then F is valid. If one of them is valid, and the other is equivalent to x_0, then $F \equiv x_0$.

Case $k + 1$ We have the following induction hypothesis (IH_0): Any formula φ with $\max\{i \in \mathbb{N} \mid x_i \in \mathrm{Vars}(\varphi)\} \leq k$ is either valid or equivalent to a formula in normal form.

We now prove this case by structural induction on F:

Case $F = x_j$ Then trivially $F \equiv x_j$.

Case $F = G \leftrightarrow H$ We have two induction hypotheses:

IH_G: G is valid, or $G \equiv G'$ in normal form.

IH_H: F is valid, or $H \equiv H'$ in normal form.

If G is valid then $F \equiv H$ and the case follows from IH_H. Analogous if H is valid. If neither are valid then $F \equiv G' \leftrightarrow H'$ in normal form. We perform a case analysis on G' and H':

- If $G' = x_i \leftrightarrow G''$ and $H' = x_j \leftrightarrow H''$ then one of the following cases hold:
  - $i = j$ and $F \equiv (x_i \leftrightarrow x_j) \leftrightarrow (G'' \leftrightarrow H'') \equiv G'' \leftrightarrow H''$ by associativity and commutativity. This formula no longer contains x_{k+1} and by IH_0 it is either valid or has an equivalent formula in normal form.
  - $i < j$ then $F \equiv x_j \leftrightarrow (x_i \leftrightarrow (G'' \leftrightarrow H''))$. By IH_0 $(x_i \leftrightarrow (G'' \leftrightarrow H''))$ is either valid - then $F \equiv x_j$ - or equivalent to a formula F' in normal form. then $F \equiv x_j \leftrightarrow F'$
- All other cases are analogous to one of the two above.

(c) By (b) F is either valid or equivalent to F' in normal form. In the first case we are done. In the second, we prove that F' is satisfied by exactly half of all assignments

Case $F' = x$ Then F is satisfied by the assignment $x \mapsto 1$ and not by $x \mapsto 0$

Case $F' = x \leftrightarrow G$ with G in normal form and $x \notin \text{Vars}(G)$ and G. For any assignment $\mathcal{A} \in 2^{\text{Vars}(G) \cup \{x\}}$ we can define $\mathcal{A}'$ which agrees with $\mathcal{A}$ on all variables except x. Since x does not occur in G we have $\mathcal{A}(G) = \mathcal{A}'(G)$ and therfore

$$\begin{array}{ll} & \mathcal{A}(F') = 1 \\ \textsf{iff} & \mathcal{A}(x) = \mathcal{A}(G) \\ \textsf{iff} & \mathcal{A}'(x) \neq \mathcal{A}'(G) \\ \textsf{iff} & \mathcal{A}'(F') = 0 \end{array}$$

This proves that exactly half of all assignments satisfy F' and hence also F.