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Note:

- Cross your Registration number(with leading zero). It will be evaluated automatically.
- Sign in the corresponding signature field.

Logik

Exam: IN2049 / Endterm

Date: Wednesday 26th July, 2023

Examiner: Prof. Dr. Javier Esparza

Time: 13:30 – 15:30

	P 1	P 2	P 3	P 4	P 5	P 6	P 7	P 8
I								
II								

Working instructions

- This exam consists of **20 pages** with a total of **8 problems**.
Please make sure now that you received a complete copy of the exam.
- The total amount of achievable credits in this exam is 46 credits. Among them, 6 are bonus credits. In order to pass the exam, you will need at least 17 credits.
- Detaching pages from the exam is prohibited.
- Allowed resources: a single hand-written cheat-sheet (you can write on both sides of the sheet)
- **Answers are only accepted if the solution approach is documented.** Give a reason for each answer unless explicitly stated otherwise in the respective subproblem.
- Do not write with red or green colors nor use pencils.
- Physically turn off all electronic devices, put them into your bag and close the bag.

Left room from _____ to _____ / Early submission at _____



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Problem 1 Syntax of propositional logic (10 credits)

Recall that the syntax of propositional logic contains five logical operators: \neg (negation), \wedge (conjunction), \vee (disjunction), \rightarrow (implication) and \leftrightarrow (bi-implication). **All the formulas in this problem are formulas over propositional logic.**

In every subproblem, you will be asked to prove or disprove a given claim. If the claim is true, give a proof. If the claim is false, give a counter-example and prove that it does not satisfy the claim.

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a) Prove or disprove: For every formula F , there is an equivalent formula G which contains only \vee and \wedge as its logical operators.





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b) Prove or disprove: For every formula F , there is an equivalent formula G which contains only \neg and \rightarrow as its logical operators.

c) Prove or disprove: For every formula F , there is an equivalent formula G which contains only \leftrightarrow as its logical operator.

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d) **Bonus subproblem.** Prove or disprove: For every formula F , there is an equivalent formula G which contains only \neg and \leftrightarrow as its logical operators.

Hint: Consider the parity of the number of satisfying assignments.





Problem 2 Sequent calculus (3 credits)

Give a proof tree using the rules of sequent calculus to prove that the following formula over propositional logic is a tautology.

$$(((P \wedge Q) \rightarrow R)) \rightarrow ((P \rightarrow R) \vee (Q \rightarrow R)))$$

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Problem 3 Structures (6 credits)

Fix a signature $\tau = \{P, Q\}$ where P is a ternary relation symbol, and Q is a unary relation symbol. Consider the following formulas:

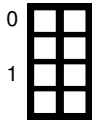
$$\varphi_1 = \forall x \exists y (P(x, x, y) \wedge \neg P(x, y, y))$$

$$\varphi_2 = \exists z \neg Q(z)$$

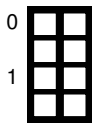
$$\varphi_3 = \forall x \forall y \exists z (P(x, z, y) \vee Q(z))$$

In each subproblem of this exercise, you will be asked to give a τ -structure satisfying some constraints. For each subproblem, if your solution is a τ -structure \mathcal{E} , present it in the following format:

- $\mathcal{U}^{\mathcal{E}} :=$
- $P^{\mathcal{E}} :=$
- $Q^{\mathcal{E}} :=$

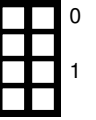


a) Construct a τ -structure \mathcal{A} such that $\mathcal{A} \models \varphi_1$, $\mathcal{A} \models \varphi_2$ and $\mathcal{A} \models \varphi_3$.

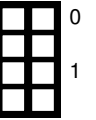


b) Construct a τ -structure \mathcal{B} such that $\mathcal{B} \models \varphi_1$, $\mathcal{B} \models \varphi_2$ and $\mathcal{B} \not\models \varphi_3$.





c) Construct a τ -structure \mathcal{C} such that $\mathcal{C} \not\models \varphi_1$, $\mathcal{C} \models \varphi_2$ and $\mathcal{C} \models \varphi_3$.



d) Construct a τ -structure \mathcal{D} such that $\mathcal{D} \not\models \varphi_1$, $\mathcal{D} \not\models \varphi_2$ and $\mathcal{D} \models \varphi_3$.





Problem 4 First order logic modeling (10 credits)

Consider the following statements.

- (S1) If a person has read somebody's work, then it was taught by somebody.
- (S2) All wise philosophers have read Aristotle's work.
- (S3) No philosopher is unwise.
- (S4) There exists at least one philosopher.
- (S5) Anybody who has taught somebody's work is a philosopher.
- (S6) No wise person has taught Aristotle's work.

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a) Formalize each of the statements S1, S2, S3, S4, S5 and S6 as closed formulas in first-order logic without equality. The only predicate symbols that you are allowed to use are two unary predicates P , W and two binary predicates R and T . The only constant symbol you can use is a .

$P(x)$ must be used to denote that x is a philosopher, $W(x)$ must be used to denote that x is wise, $R(x, y)$ must be used to denote that x has read y 's work and $T(x, y)$ must be used to denote that x has taught y 's work. a must be used for Aristotle.





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b) For every $i \in \{1, 2, 3, 4, 5, 6\}$, let F_i be the formalization of the statement S_i in first-order logic without equality that you had obtained in the previous subproblem. Let $F = F_1 \wedge F_2 \wedge F_3 \wedge F_4 \wedge F_5 \wedge F_6$. Note that F is a closed formula. Convert F into an equivalent rectified formula in prenex form F' .





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c) Consider the formula F' from the previous subproblem. Convert F' into an equisatisfiable closed formula in Skolem form G . **The matrix of the formula G must be in CNF where each clause is a Horn clause.**





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d) Consider the formula G from the previous subproblem. Construct a finite unsatisfiable subset K of the clause Herbrand expansion of G . You have to prove that K is not satisfiable by using the Horn satisfiability algorithm.

Hence this indicates that the collection of statements S_1, S_2, S_3, S_4, S_5 and S_6 is unsatisfiable.





Problem 5 Herbrand theory (5 credits)

Let \mathcal{A} be the structure defined as follows:

$$\begin{aligned} U_{\mathcal{A}} &= \mathbb{N} \setminus \{0\} \\ c^{\mathcal{A}} &= 5 \\ f^{\mathcal{A}}(x, y) &= x + y \\ (m, n) \in P^{\mathcal{A}} &\Leftrightarrow m < n \\ n \in Q^{\mathcal{A}} &\Leftrightarrow n \text{ is divisible by } 10 \end{aligned}$$

Let φ be the following formula:

$$\forall x \forall y \quad P(x, f(x, y)) \wedge \neg P(f(x, y), y) \wedge Q(f(c, c)).$$

Note that $\mathcal{A} \models \varphi$. Using the construction from the Fundamental theorem of predicate logic, construct a Herbrand structure \mathcal{H} that is a model for φ based on \mathcal{A} .

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Problem 6 Predicate logic resolution (3 credits)

Let c and d be constant symbols, let P be a unary predicate symbol and let Q be a ternary predicate symbol. Consider the following clauses:

$$C_1 = \{\neg Q(c, x, y), P(g(x))\}$$

$$C_2 = \{Q(x, f(x), g(y))\}$$

$$C_3 = \{\neg Q(x, f(d), y), \neg P(y)\}$$

Use the predicate logic resolution to prove unsatisfiability of $C_1 \wedge C_2 \wedge C_3$. In each step explain which clauses you are considering, what is their most general unifier and what is their resolvent.

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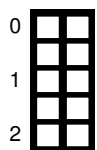




Problem 7 Compactness and completeness (5 credits)



a) Suppose Γ is a set of sentences in first-order logic with equality. For every $m \in \mathbb{N}$, construct a formula φ_m such that \mathcal{A} is a model for $\Gamma \cup \{\varphi_m\}$ if and only if \mathcal{A} is a model for Γ and the size of the universe of \mathcal{A} is at least m .



b) Suppose Γ is a set of sentences in first-order logic with equality such that for every $m \in \mathbb{N}$, Γ has a model whose universe has at least m elements. Prove that Γ has a model with an infinite universe.

Hint: You are allowed to use the formulas $\{\varphi_m : m \in \mathbb{N}\}$ from the previous subproblem.





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c) **Bonus subproblem.** Let T be a theory in first-order logic. We have shown in the lectures that if all models of T are elementary equivalent, then T is complete. Prove the following stronger statement here: If all *countable models* of T are elementary equivalent, then T is complete.





Problem 8 Consistency (4 credits)

Suppose T is a theory in first-order logic. T is said to be *consistent* if for every sentence S , it **does not** include both S and $\neg S$.

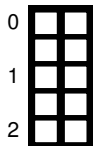
Consider the signature $\Sigma = \{+, *, \leq\}$. Recall that $(\mathbb{Z}, +, *, \leq)$ denotes the structure with universe \mathbb{Z} and the standard interpretations for the symbols $+$, $*$ and \leq .



a) Give an example of a Σ -theory which is complete, decidable, contains $Th(\mathbb{Z}, +, *, \leq)$, and is not consistent.



b) Give an example of a Σ -theory which is complete, consistent, contains $Th(\mathbb{Z}, +, *, \leq)$, but is not decidable.



c) Give an example of a Σ -theory which is complete, decidable, consistent, but does not contain $Th(\mathbb{Z}, +, *, \leq)$.



Additional space for solutions—clearly mark the (sub)problem your answers are related to and strike out invalid solutions.

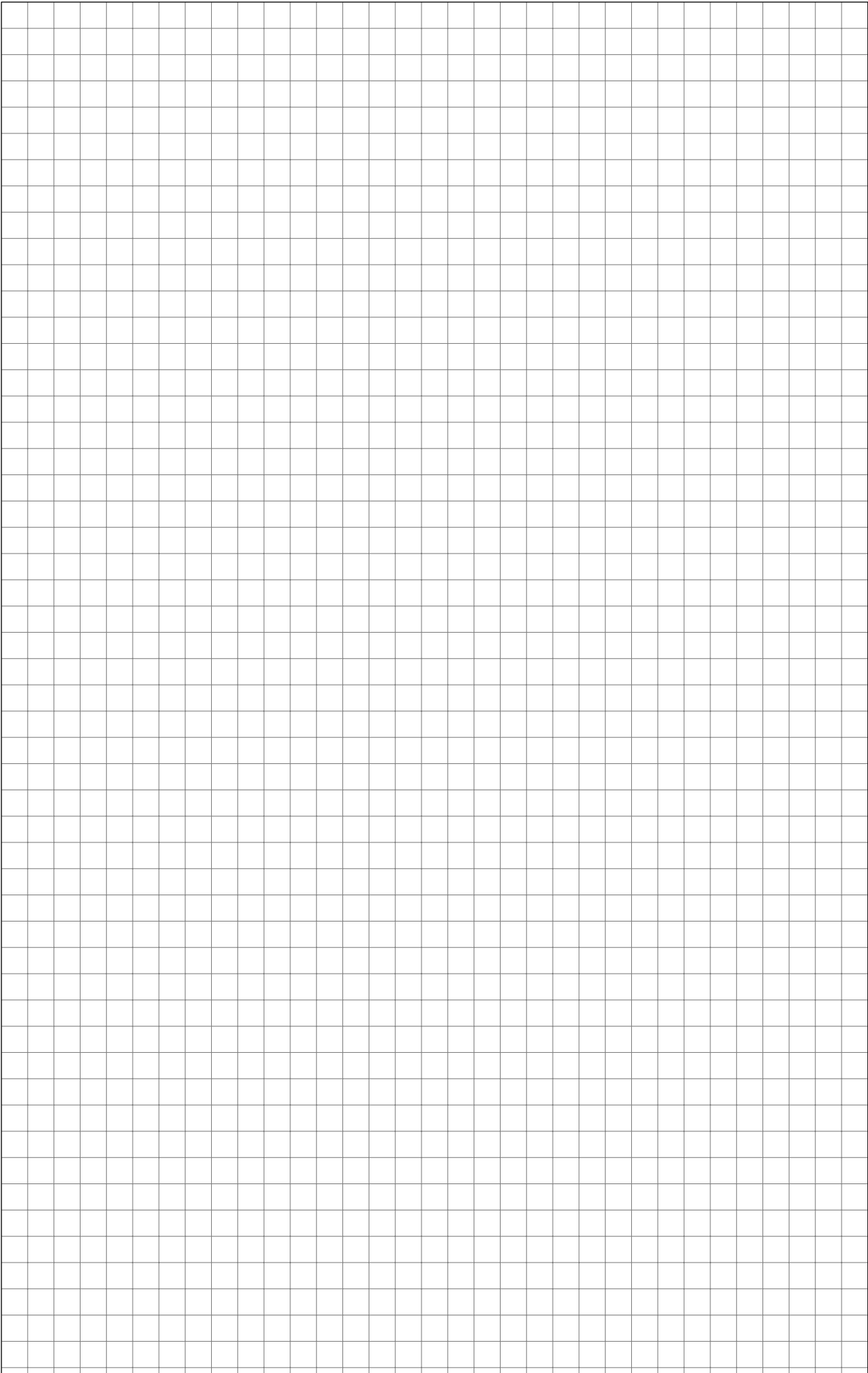


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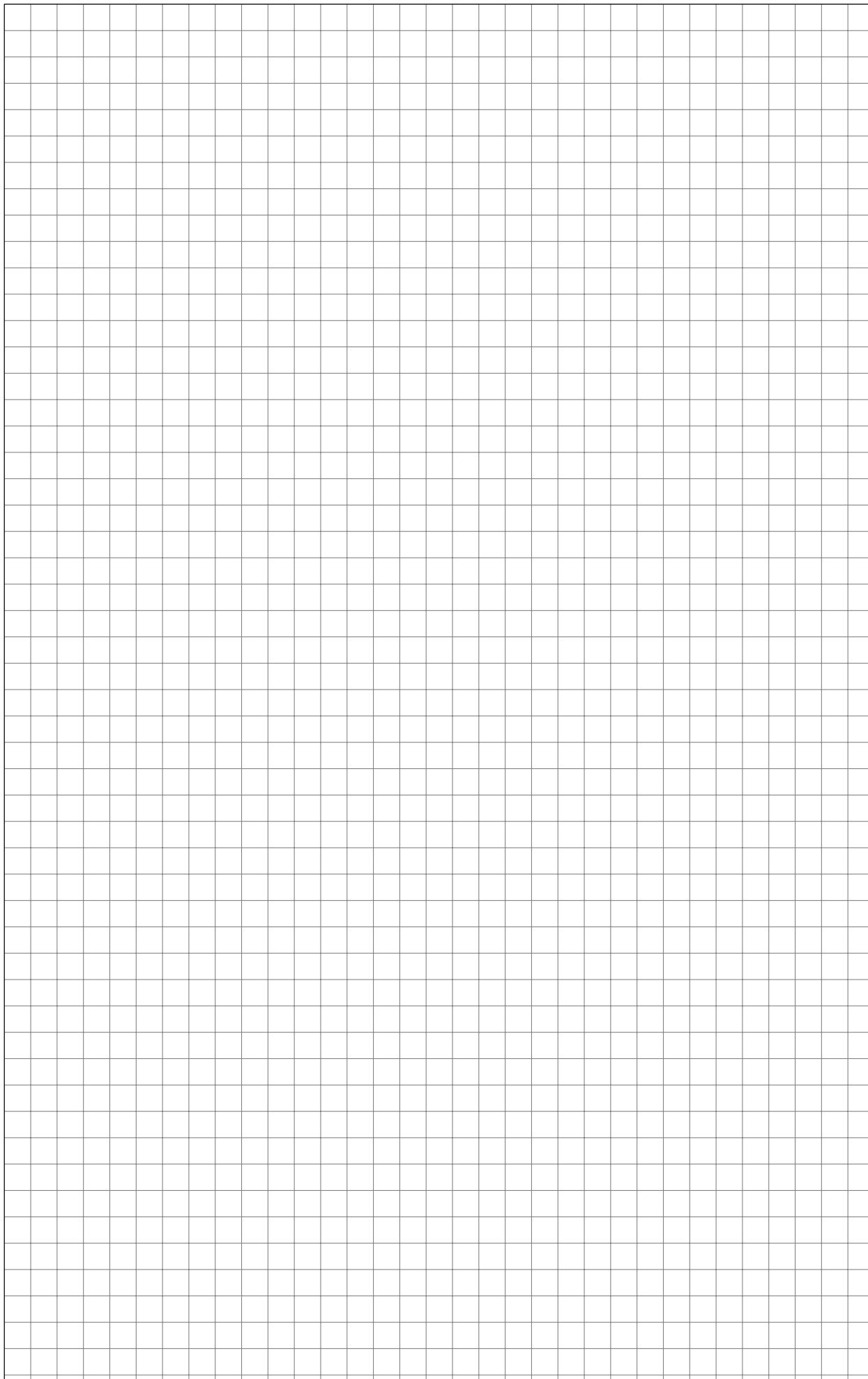
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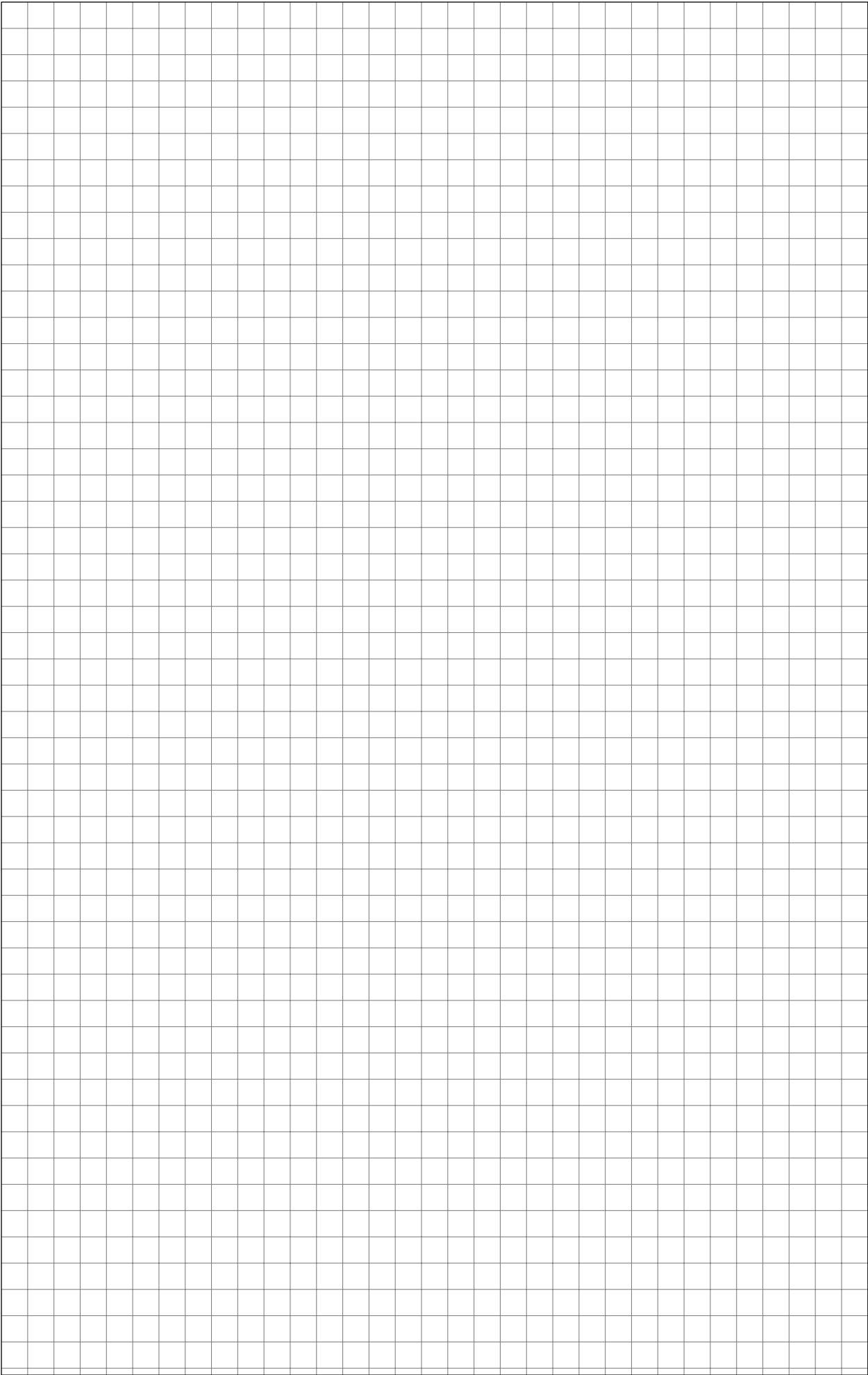




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