This is the exam for the Logic course that happened during the summer semester of 2020. Some of the questions were assigned to the students randomly; no student received all of these questions.

Question 1: Satisfiability and normal forms in first order logic

Let F be the following formula:

$$\big(\forall x R(x,x) \wedge \forall x \forall y (R(x,y) \to R(y,x))\big) \to \forall x \forall y \forall z (R(x,y) \wedge R(y,z) \to R(x,z))$$

1. Give a structure $\mathcal{A}$ such that $\mathcal{A} \models F$.
2. Give a structure $\mathcal{B}$ such that $\mathcal{B} \models \neg F$.
3. Give a formula in rectified form which is equivalent to F.
4. Give a formula in prenex form which is equivalent to F.
5. Give a formula in Skolem form which is equivalent to F.

Solutions

1. Let us define $\mathcal{A}$ such that $U_{\mathcal{A}} = \{a, b\}$ and $R^{\mathcal{A}} = \{(a,a)\}$. Then $\mathcal{A} \models F$.
2. Let $U_{\mathcal{B}} = \{a,b,c\}$ and $R^{\mathcal{B}} = \{(a,a),(b,b),(c,c),(a,b),(b,a),(b,c),(c,b)\}$. Then $\mathcal{B} \models \neg F$.
3. $$F \equiv \big(\forall x R(x,x) \wedge \forall u \forall y (R(u,y) \to R(y,u))\big) \to \forall t \forall v \forall z (R(t,v) \wedge R(v,z) \to R(t,z))$$
4. $$\begin{aligned} F &\equiv \big(\forall x R(x,x) \wedge \forall u \forall y (R(u,y) \to R(y,u))\big) \to \forall t \forall v \forall z (R(t,v) \wedge R(v,z) \to R(t,z)) \\ &\equiv \neg\big(\forall x R(x,x) \wedge \forall u \forall y (R(u,y) \to R(y,u))\big) \vee \forall t \forall v \forall z (R(t,v) \wedge R(v,z) \to R(t,z)) \\ &\equiv \big(\neg\forall x R(x,x) \vee \neg\forall u \forall y (R(u,y) \to R(y,u))\big) \vee \forall t \forall v \forall z (R(t,v) \wedge R(v,z) \to R(t,z)) \\ &\equiv \exists x \neg R(x,x) \vee \exists u \exists y \neg(R(u,y) \to R(y,u)) \vee \forall t \forall v \forall z (R(t,v) \wedge R(v,z) \to R(t,z)) \\ &\equiv \exists x \exists u \exists y \forall t \forall v \forall z \big(\neg R(x,x) \vee \neg(R(u,y) \to R(y,u)) \vee (R(t,v) \wedge R(v,z) \to R(t,z))\big) \end{aligned}$$
5. $$\begin{aligned} F &\equiv \exists x \exists u \exists y \forall t \forall v \forall z \big(\neg R(x,x) \vee \neg(R(u,y) \to R(y,u)) \vee (R(t,v) \wedge R(v,z) \to R(t,z))\big) \\ &\equiv_s \exists u \exists y \forall t \forall v \forall z \big(\neg R(c,c) \vee \neg(R(u,y) \to R(y,u)) \vee (R(t,v) \wedge R(v,z) \to R(t,z))\big) \\ &\equiv_s \exists y \forall t \forall v \forall z \big(\neg R(c,c) \vee \neg(R(d,y) \to R(y,d)) \vee (R(t,v) \wedge R(v,z) \to R(t,z))\big) \\ &\equiv_s \forall t \forall v \forall z \big(\neg R(c,c) \vee \neg(R(d,e) \to R(e,d)) \vee (R(t,v) \wedge R(v,z) \to R(t,z))\big) \end{aligned}$$

Question 2: Satisfiability and normal forms in first order logic

Let F be the following formula:

$$\big(\forall x R(x,x) \wedge \forall x \forall y \forall z (R(x,y) \wedge R(y,z) \to R(x,z))\big) \to \forall x \forall y (R(x,y) \to R(y,x))$$

1. Give a structure $\mathcal{A}$ such that $\mathcal{A} \models F$.
2. Give a structure $\mathcal{B}$ such that $\mathcal{B} \models \neg F$.
3. Give a formula in rectified form which is equivalent to F.
4. Give a formula in prenex form which is equivalent to F.
5. Give a formula in Skolem form which is equivalent to F.

Solutions

1. Let us define $\mathcal{A}$ such that $U_{\mathcal{A}} = \{a, b\}$ and $R^{\mathcal{A}} = \{(a,a)\}$. Then $\mathcal{A} \models F$.
2. Let $U_{\mathcal{B}} = \{a, b\}$ and $R^{\mathcal{B}} = \{(a,a), (b,b), (a,b)\}$. Then $\mathcal{B} \models \neg F$.
3. $$\big(\forall x R(x,x) \wedge \forall t \forall y \forall z (R(t,y) \wedge R(y,z) \to R(t,z))\big) \to \forall u \forall v (R(u,v) \to R(v,u))$$
4. $$\begin{aligned} F &\equiv \big(\forall x R(x,x) \wedge \forall t \forall y \forall z (R(t,y) \wedge R(y,z) \to R(t,z))\big) \to \forall u \forall v (R(u,v) \to R(v,u)) \\ &\equiv \neg\big(\forall x R(x,x) \wedge \forall t \forall y \forall z (R(t,y) \wedge R(y,z) \to R(t,z))\big) \vee \forall u \forall v (R(u,v) \to R(v,u)) \\ &\equiv \big(\neg \forall x R(x,x) \vee \neg \forall t \forall y \forall z (R(t,y) \wedge R(y,z) \to R(t,z))\big) \vee \forall u \forall v (R(u,v) \to R(v,u)) \\ &\equiv \exists x \neg R(x,x) \vee \exists t \exists y \exists z \neg (R(t,y) \wedge R(y,z) \to R(t,z)) \vee \forall u \forall v (R(u,v) \to R(v,u)) \\ &\equiv \exists x \exists t \exists y \exists z \forall u \forall v \big(\neg R(x,x) \vee \neg (R(t,y) \wedge R(y,z) \to R(t,z)) \vee (R(u,v) \to R(v,u))\big) \end{aligned}$$
5. $$\begin{aligned} F &\equiv \exists x \exists t \exists y \exists z \forall u \forall v \big(\neg R(x,x) \vee \neg (R(t,y) \wedge R(y,z) \to R(t,z)) \vee (R(u,v) \to R(v,u))\big) \\ &\equiv_s \exists t \exists y \exists z \forall u \forall v \big(\neg R(c,c) \vee \neg (R(t,y) \wedge R(y,z) \to R(t,z)) \vee (R(u,v) \to R(v,u))\big) \\ &\equiv_s \exists y \exists z \forall u \forall v \big(\neg R(c,c) \vee \neg (R(d,y) \wedge R(y,z) \to R(d,z)) \vee (R(u,v) \to R(v,u))\big) \\ &\equiv_s \exists z \forall u \forall v \big(\neg R(c,c) \vee \neg (R(d,e) \wedge R(e,z) \to R(d,z)) \vee (R(u,v) \to R(v,u))\big) \\ &\equiv_s \forall u \forall v \big(\neg R(c,c) \vee \neg (R(d,e) \wedge R(e,f) \to R(d,f)) \vee (R(u,v) \to R(v,u))\big) \end{aligned}$$

Question 3: Satisfiability and normal forms in first order logic

Let F be the following formula:

$$\big(\forall x \forall y (R(x,y) \rightarrow R(y,x)) \wedge \forall x \forall y \forall z (R(x,y) \wedge R(y,z) \rightarrow R(x,z))\big) \rightarrow \forall x R(x,x)$$

1. Give a structure $\mathcal{A}$ such that $\mathcal{A} \models F$.
2. Give a structure $\mathcal{B}$ such that $\mathcal{B} \models \neg F$.
3. Give a formula in rectified form which is equivalent to F.
4. Give a formula in prenex form which is equivalent to F.
5. Give a formula in Skolem form which is equivalent to F.

Solutions

1. Let us define $\mathcal{A}$ such that $U_{\mathcal{A}} = \{a, b\}$ and $R^{\mathcal{A}} = \{(a,b)\}$. Then $\mathcal{A} \models F$.
2. Let $U_{\mathcal{B}} = \{a, b\}$ and $R^{\mathcal{B}} = \{(a,a)\}$. Then $\mathcal{B} \models \neg F$.
3. $$F \equiv \big(\forall x \forall y (R(x,y) \rightarrow R(y,x)) \wedge \forall t \forall u \forall z (R(t,u) \wedge R(u,z) \rightarrow R(t,z))\big) \rightarrow \forall v R(v,v)$$
4. $$\begin{aligned} F &\equiv \big(\forall x \forall y (R(x,y) \rightarrow R(y,x)) \wedge \forall t \forall u \forall z (R(t,u) \wedge R(u,z) \rightarrow R(t,z))\big) \rightarrow \forall v R(v,v) \\ &\equiv \neg\big(\forall x \forall y (R(x,y) \rightarrow R(y,x)) \wedge \forall t \forall u \forall z (R(t,u) \wedge R(u,z) \rightarrow R(t,z))\big) \vee \forall v R(v,v) \\ &\equiv \big(\neg \forall x \forall y (R(x,y) \rightarrow R(y,x)) \vee \neg \forall t \forall u \forall z (R(t,u) \wedge R(u,z) \rightarrow R(t,z))\big) \vee \forall v R(v,v) \\ &\equiv \exists x \exists y \neg (R(x,y) \rightarrow R(y,x)) \vee \exists t \exists u \exists z \neg (R(t,u) \wedge R(u,z) \rightarrow R(t,z))\big) \vee \forall v R(v,v) \\ &\equiv \exists x \exists y \exists t \exists u \exists z \forall v \big(\neg (R(x,y) \rightarrow R(y,x)) \vee \neg (R(t,u) \wedge R(u,z) \rightarrow R(t,z))\big) \vee R(v,v)\big) \end{aligned}$$
5. $$F \equiv_s \forall v \big(\neg (R(c,d) \rightarrow R(d,c)) \vee \neg (R(e,f) \wedge R(f,g) \rightarrow R(e,g))\big) \vee R(v,v)\big)$$

Question 4: Propositional formulas

We are given the following data about one technical university in Krakozhia:

- Creative benevolent logicians are gamblers.
- Logicians are either creative or do not play chess.
- Smokers who wear ties do not gamble.
- Those who do not play chess are not logicians.
- Those who are benevolent and creative, are smokers.

1. Formalize the statements above in propositional logic.
2. Use resolution to conclude that benevolent logicians do not wear ties.

Question 5: Propositional formulas

We are given the following data about one generation of students in Hogwarts:

- Those who are hard-working and witty, are friendly.
- Muggle-borns are either witty or do not play quidditch.
- Those who are friendly and have owls, are not patient.
- Witty hard-working muggle-borns are patient.
- Those who do not play quidditch are not muggle-born.

1. Formalize the statements above in propositional logic.
2. Use resolution to conclude that hard-working muggle-born do not have owls.

Question 6: Propositional formulas

The logic team is up to no good and plans a bank robbery. The following information is known.

- If Marijana agrees with the plan and the equipment is bought and Christoph plans the bank robbery then the bank robbery succeeds.
- If Christoph plans the bank robbery then either Marijana agrees with the plan or Professor Evilsparza does not participate.
- If Inspector Luttenberger gets a tip-off and there is an error in planing then the bank robbery fails.
- If Professor Evilsparza does not participate then Christoph does not plan the bank robbery.

- If Marijana agrees with the plan and the equipment is bought then Inspector Luttenberger gets a tip-off.

1. Formalize the statements above in propositional logic.
2. Use resolution to conclude that if Christoph plans the bank robbery and the equipment is bought then there is no error in planing.

Question 7: Propositional formulas

Middle-earth is in danger.

- If Aragorn rallies the dead and Boromir sacrifices himself for the Hobbits and Frodo destroys the One Ring then Sauron is defeated.
- If Frodo destroys the One Ring then either Aragon rallies the dead or Gollum conspires with Shelob.
- If Gandalf becomes the White but the Corsairs of Umbar reach Pellenor Fields then Sauron is not defeated.
- If Gollum conspires with Shelob then Frodo does not destroy the One Ring.
- If Aragon rallies the dead and Boromir sacrifices himself for the Hobbits then Gandalf becomes the White.

1. Formalize the statements above in propositional logic.
2. Use resolution to conclude that if Frodo destroys the One Ring and Boromir sacrifices himself for the Hobbits then the Corsairs of Umbar do not reach Pellenor Fields.

Solutions

Let us introduce the following notation:

- A = creative
- B = benevolent
- C = logician
- D = gambler
- E = do not play chess
- F = smoker
- G = wear ties

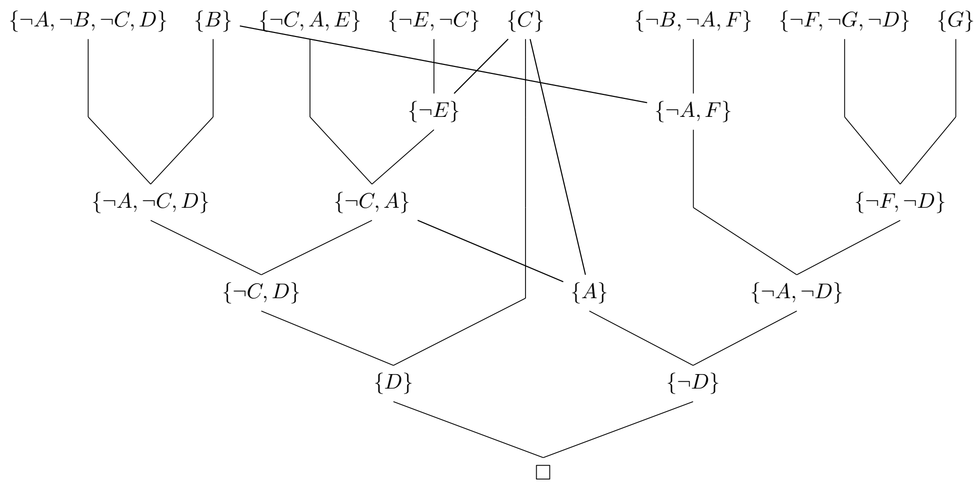
We can formalize the statements in propositional logic as follows:

- $A \wedge B \wedge C \rightarrow D \equiv \neg A \vee \neg B \vee \neg C \vee D$
- $C \rightarrow A \vee E \equiv \neg C \vee A \vee E$
- $F \wedge G \rightarrow \neg D \equiv \neg F \vee \neg G \vee \neg D$
- $E \rightarrow \neg C \equiv \neg E \vee \neg C$
- $B \wedge A \rightarrow F \equiv \neg B \vee \neg A \vee F$

The conclusion can be formalized as $B \wedge C \rightarrow \neg G$.

In order to prove that benevolent logicians do not wear ties, we show that assuming the opposite ($B \wedge C \wedge G$) will lead to a contradiction. That is, we prove by resolution that the following CNF is unsatisfiable:

$$\{\{\neg A, \neg B, \neg C, D\}, \{\neg C, A, E\}, \{\neg F, \neg G, \neg D\}, \{\neg E, \neg C\}, \{\neg B, \neg A, F\}, \{B\}, \{C\}, \{G\}\}$$



The solution is the same for other versions of this exercise, we just introduce different notation.

For the students of Hogwarts:

- A = witty
- B = hard-working
- C = muggle-born
- D = patient
- E = do not play quidditch
- F = friendly

- G = have owls

For the bank robbery:

- A = Marijana agrees with the plan
- B = the equipment is bought
- C = Christoph plans the bank robbery
- D = the bank robbery succeeds
- E = Professor Evilsparza does not participate
- F = Inspector Uwe Schöning gets a tip-off
- G = there is an error in planing

For the middle-earth crises:

- A = Aragorn rallies the dead
- B = Boromir sacrifices himself for the Hobbits
- C = Frodo destroys the One Ring
- D = Sauron is defeated
- E = Gollum conspires with Shelob
- F = Gandalf becomes the White
- G = the Corsairs of Umbar reach Pellenor Fields

Question 8: Propositional formulas

Let F be a propositional formula. Then, we set

$$A_F = F \wedge (p \vee \neg q) \qquad\qquad B_F = \neg q \vee F$$

Find F such that $A_F \equiv B_F$.

Solutions

$$\varphi = \neg(\neg p \wedge q) \wedge z \qquad\qquad \psi = \neg q \vee z$$

p	q	z	$\neg(\neg p \wedge q) \wedge z$	$\neg q \vee z$
0	0	0	0	1
0	0	1	1	1
0	1	0	0	0
0	1	1	0	1
1	0	0	0	1
1	0	1	1	1
1	1	0	0	0
1	1	1	1	1

We define $\eta = q \to p$. Then we have

$$\begin{aligned} \varphi[\eta/z] &= \neg(\neg p \wedge q) \wedge (q \to p) \\ &\equiv (p \vee \neg q) \wedge (p \vee \neg q) \\ &\equiv p \vee \neg q \\ &\equiv \neg q \vee (\neg q \vee p) \\ &\equiv \neg q \vee (q \to p) = \psi[\eta/z] \end{aligned}$$

Question 9: Propositional formulas

Let F be a propositional formula. Then, we set

$$A_F = F \wedge \neg(\neg r \wedge p) \qquad\qquad B_F = p \to F$$

Find F such that $A_F \equiv B_F$.

Solutions

$$\varphi = (r \vee \neg p) \wedge z \qquad\qquad \psi = p \to z$$

r	p	z	$(r \vee \neg p) \wedge z$	$p \to z$
0	0	0	0	1
0	0	1	1	1
0	1	0	0	0
0	1	1	0	1
1	0	0	0	1
1	0	1	1	1
1	1	0	0	0
1	1	1	1	1

We define $\eta = p \rightarrow r$. Then we have

$$\begin{aligned}\varphi[\eta/z] &= (r \vee \neg p) \wedge (p \rightarrow r) \\ &\equiv (\neg p \vee r) \wedge (\neg p \vee r) \\ &\equiv \neg p \vee r \\ &\equiv \neg p \vee (\neg p \vee r) \\ &\equiv p \rightarrow (p \rightarrow r) = \psi[\eta/z]\end{aligned}$$

Question 10: Propositional formulas

Let F be a propositional formula. Then, we set

$$A_F = F \wedge (t \rightarrow s) \qquad\qquad B_F = \neg F \rightarrow \neg t$$

Find F such that $A_F \equiv B_F$.

Solutions

$$\varphi = (t \rightarrow s) \wedge z \qquad\qquad \psi = t \rightarrow z$$

t	s	z	$(t \rightarrow s) \wedge z$	$t \rightarrow z$
0	0	0	0	1
0	0	1	1	1
0	1	0	0	1
0	1	1	1	1
1	0	0	0	0
1	0	1	0	1
1	1	0	0	0
1	1	1	1	1

We define $\eta = t \rightarrow s$. Then we have

$$\begin{aligned}\varphi[\eta/z] &= (t \rightarrow s) \wedge (t \rightarrow s) \\ &\equiv t \rightarrow s \\ &\equiv \neg t \vee s \\ &\equiv \neg t \vee (\neg t \vee s) \\ &\equiv \neg t \vee (t \rightarrow s) \\ &\equiv t \rightarrow (t \rightarrow s) = \psi[\eta/z]\end{aligned}$$

Question 11: Propositional proof

Let $\varphi_n = X_1 \leftrightarrow X_2 \leftrightarrow \ldots \leftrightarrow X_n$ for some $n \in \mathbb{N}$. Prove that for every $\mathcal{I}\colon \{X_1, \ldots, X_n\} \rightarrow \{0, 1\}$ holds

$$\mathcal{I} \models \varphi_n \text{ if and only if } \mathcal{I}[A \mapsto \overline{\mathcal{I}(A)}] \not\models \varphi_n \text{ for all } A \in \mathbb{X}.$$

Question 12: Propositional proof

Let $\varphi_n = X_1 \oplus X_2 \oplus \ldots \oplus X_n$ for some $n \in \mathbb{N}$. Prove that for every $\mathcal{I}\colon \{X_1, \ldots, X_n\} \to \{0,1\}$ holds

$$\mathcal{I} \models \varphi_n \text{ if and only if } \mathcal{I}[A \mapsto \overline{\mathcal{I}(A)}] \not\models \varphi_n \text{ for all } A \in \mathbb{X}.$$

Question 13: Propositional proof

Let $\varphi_1 = X_1$ and $\varphi_{n+1} = \varphi_n \leftrightarrow X_{n+1}$ for all $n > 1$.

Prove that for every φ_n there are the same number of satisfying and non-satisfying interpretations (w.r.t. the variables $\{X_1, \ldots, X_n\}$). More formally, prove that

$$|\{\mathcal{I}\colon \{X_1, \ldots, X_n\} \to \{0,1\} \mid \mathcal{I} \models \varphi_n\}| = |\{\mathcal{I}\colon \{X_1, \ldots, X_n\} \to \{0,1\} \mid \mathcal{I} \not\models \varphi_n\}|.$$

Solution

Let us first introduce some simplifying notation. For example, let us denote the set of all satisfying assignments of φ_n by A_n^1, and the set of all non-satisfying assignments of φ_n by A_n^0:

$$\begin{aligned} A_n^1 &= \{\mathcal{I}\colon \{X_1, \ldots, X_n\} \to \{0,1\} \mid \mathcal{I} \models \varphi_n\} \\ A_n^0 &= \{\mathcal{I}\colon \{X_1, \ldots, X_n\} \to \{0,1\} \mid \mathcal{I} \models \varphi_n\} \end{aligned}$$

We prove that the number of satisfying assignments is

$$\left|A_n^1\right| = 2^{n-1}.$$

Since the set of all assignments with n variables is $\{\mathcal{I}\colon \{X_1, \ldots, X_n\} \to \{0,1\}\} = A_n^1 \cup A_n^0$, and since its size is $|\{\mathcal{I}\colon \{X_1, \ldots, X_n\} \to \{0,1\}\}| = 2^n$, this implies that the number of non-satisfying assignments is the same,

$$\left|A_n^0\right| = 2^n - \left|A_n^1\right| = 2^n - 2^{n-1} = 2^{n-1} = \left|A_n^1\right|.$$

Therefore, it is enough to prove that for every $n \geq 1$ we have $\left|A_n^1\right| = 2^{n-1}$. We prove this claim by induction on n.

Induction base. If $n = 1$ there is exactly $1 = 2^{1-1}$ assignment that satisfies $\varphi_1 = X_1$, namely $\mathcal{I}\colon \{X_1\} \to \{0,1\}$ with $\mathcal{I}(X_1) = 1$.

Induction hypothesis. Let us assume the claim holds for n, that is, we assume $\left|A_n^1\right| = 2^{n-1}$, which also implies that $\left|A_n^0\right| = 2^{n-1}$.

Induction step. Let us now prove the statement for $n+1$, that is, we show that

$$\left|A_{n+1}^1\right| = 2^n.$$

Note that $\mathcal{I} \in A_{n+1}^1$ if and only if $\mathcal{I}\colon \{X_1, \ldots, X_n, X_{n+1}\} \to \{0,1\}$ and $\mathcal{I} \models \varphi_{n+1}$. Since $\varphi_{n+1} = \varphi_n \leftrightarrow X_{n+1}$, we also have that $\mathcal{I} \models \varphi_{n+1}$ if and only if one of the following holds:

- $\mathcal{I} \models \varphi_n$ and $\mathcal{I} \models X_{n+1}$, or
- $\mathcal{I} \not\models \varphi_n$ and $\mathcal{I} \not\models X_{n+1}$.

In other words, for an assignment $\mathcal{I}\colon \{X_1, \ldots, X_n, X_{n+1}\} \to \{0,1\}$ we have $\mathcal{I} \in A^1_{n+1}$ if and only if one of the following holds:

- $\mathcal{I}|_{X_1,\ldots,X_n} \in A^1_n$ and $\mathcal{I}(X_{n+1}) = 1$, or
- $\mathcal{I}|_{X_1,\ldots,X_n} \in A^0_n$ and $\mathcal{I}(X_{n+1}) = 0$.

By the induction hypothesis, there are 2^{n-1} assignments that satisfy the first condition, and 2^{n-1} assignments that satisfy the second one. That is, there are exactly $2^{n-1} + 2^{n-1} = 2^n$ assignments in A^1_{n+1}, which proves our statement.

Question 14: Propositional proof

Let $\varphi_n = X_1 \oplus X_2 \oplus \ldots \oplus X_n$ for some $n \in \mathbb{N}$ and $\mathbb{X} = \{X_1, \ldots, X_n\}$. Prove that

$$|\{\mathcal{I}\colon \mathbb{X} \to \{0,1\} \mid \mathcal{I} \models \varphi_n\}| = |\{\mathcal{I}\colon \mathbb{X} \to \{0,1\} \mid \mathcal{I} \not\models \varphi_n\}|.$$

Question 15: Partial order with compactness

Let $\tau = \{\sim, c\}$ be a signature where $\sim$ is a binary relation symbol and c a constant symbol.

1. Give a set of formulae Φ in first-order logic with equality such that for every structure $\mathcal{A}$ holds

$$\mathcal{A} \models \Phi \text{ if and only if } \sim^{\mathcal{A}} \text{ is an equivalence relation and } [c^{\mathcal{A}}]_{\sim^{\mathcal{A}}} \text{ is infinite.}$$

   Remark

   An equivalence relation is a symmetric, transitive, and reflexive relation.

2. Prove that there is no formula φ in first-order logic with equality such that for every structure $\mathcal{A}$ holds

$$\mathcal{A} \models \varphi \text{ if and only if } \mathcal{A} \models \Phi.$$

Question 16: Simple graphs with compactness

Let $\tau = \{E, c\}$ be a signature where E is a binary relation symbol and c a constant symbol.

1. Give a set of formulae Φ in first-order logic with equality such that for every structure $\mathcal{A}$ holds

$$\mathcal{A} \models \Phi \text{ if and only if } \left\langle \mathcal{U}^{\mathcal{A}}, E^{\mathcal{A}} \right\rangle \text{ is a simple graph}$$
$$\text{and } c^{\mathcal{A}} \text{ is adjacent to infinitly many elements.}$$

Remark

A simple graph has an edge relation which is symmetric and irreflexive.

2. Prove that there is no formula φ in first-order logic with equality such that for every structure $\mathcal{A}$ holds

$$\mathcal{A} \models \varphi \text{ if and only if } \mathcal{A} \models \Phi.$$

Question 17: Partial order with compactness

Let $\tau = \{\leq, c\}$ be a signature where $\leq$ is a binary relation symbol and c a constant symbol.

1. Give a set of formulae Φ in first-order logic with equality such that for every structure $\mathcal{A}$ holds

$$\mathcal{A} \models \Phi \text{ if and only if } \leq^{\mathcal{A}} \text{ is a partial order on } \mathcal{U}^{\mathcal{A}}$$
$$\text{and } \left\langle c^{\mathcal{A}}, u \right\rangle \in \leq^{\mathcal{A}} \text{ for infinitly many } u \in \mathcal{U}^{\mathcal{A}}.$$

Remark

A partial order is a transitive, antisymmetric, and reflexive relation.

2. Prove that there is no formula φ in first-order logic with equality such that for every structure $\mathcal{A}$ holds

$$\mathcal{A} \models \varphi \text{ if and only if } \mathcal{A} \models \Phi.$$

Question 18: Herbrand structure

Consider for this exercise first-order logic *without* equality. We fix a signature $\tau = \{P, Q, f\}$ where P is a unary predicate symbol, Q is a binary predicate symbol and f is a unary function symbol. We introduce three structures $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$.

$$\mathcal{U}^{\mathcal{A}} = \{1, 2, 3, 4\}$$
$$f^{\mathcal{A}}(x) = \begin{cases} 4 & \text{if } x = 1 \\ 1 & \text{if } x = 2 \\ 1 & \text{if } x = 3 \\ 2 & \text{if } x = 4 \end{cases}$$
$$Q^{\mathcal{A}} = \{\langle 1, 3\rangle, \langle 4, 3\rangle, \langle 2, 3\rangle\}$$
$$P^{\mathcal{A}} = \{1, 3\}$$

$$\mathcal{U}^{\mathcal{B}} = \{a, b, c, d\}$$
$$f^{\mathcal{B}}(x) = \begin{cases} d & \text{if } x = a \\ b & \text{if } x = b \\ a & \text{if } x = c \\ c & \text{if } x = d \end{cases}$$
$$Q^{\mathcal{B}} = \{\langle a, c\rangle, \langle c, d\rangle, \langle d, a\rangle\}$$
$$P^{\mathcal{B}} = \{d, c\}$$

$$\mathcal{U}^{\mathcal{C}} = \{\alpha, \beta, \gamma, \eta\}$$
$$f^{\mathcal{C}}(x) = \begin{cases} \eta & \text{if } x = \alpha \\ \alpha & \text{otherwise} \end{cases}$$
$$Q^{\mathcal{C}} = \{\langle \beta, \gamma\rangle, \langle \gamma, \eta\rangle\}$$
$$P^{\mathcal{C}} = \{\beta, \gamma, \eta\}$$

1. Pick one structure from $\mathcal{A}$, $\mathcal{B}$, or $\mathcal{C}$ which satisfies the formula

$$\varphi = \exists x \forall y \forall z (P(x) \wedge \neg P(y) \to (Q(y, z) \to P(z))).$$

2. Construct the Herbrand model for φ from your choice above.

Question 19: Herbrand structure

Consider for this exercise first-order logic *without* equality. We fix a signature $\tau = \{P, Q, f\}$ where P is a unary predicate symbol, Q is a binary predicate symbol and f is a unary function symbol. We introduce three structures $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$.

$$\mathcal{U}^{\mathcal{A}} = \{1, 2, 3, 4\}$$
$$f^{\mathcal{A}}(x) = \begin{cases} 4 & \text{if } x = 1 \\ 1 & \text{if } x = 2 \\ 1 & \text{if } x = 3 \\ 2 & \text{if } x = 4 \end{cases}$$
$$Q^{\mathcal{A}} = \{\langle 1, 3\rangle, \langle 4, 3\rangle, \langle 2, 3\rangle\}$$
$$P^{\mathcal{A}} = \{1, 3\}$$

$$\mathcal{U}^{\mathcal{B}} = \{a, b, c, d\}$$
$$f^{\mathcal{B}}(x) = \begin{cases} d & \text{if } x = a \\ b & \text{if } x = b \\ a & \text{if } x = c \\ c & \text{if } x = d \end{cases}$$
$$Q^{\mathcal{B}} = \{\langle a, c\rangle, \langle c, d\rangle, \langle d, a\rangle\}$$
$$P^{\mathcal{B}} = \{d, c\}$$

$$\mathcal{U}^{\mathcal{C}} = \{\alpha, \beta, \gamma, \eta\}$$
$$f^{\mathcal{C}}(x) = \begin{cases} \eta & \text{if } x = \alpha \\ \alpha & \text{otherwise} \end{cases}$$
$$Q^{\mathcal{C}} = \{\langle \beta, \gamma\rangle, \langle \gamma, \eta\rangle\}$$
$$P^{\mathcal{C}} = \{\beta, \gamma, \eta\}$$

1. Pick one structure from $\mathcal{A}$, $\mathcal{B}$, or $\mathcal{C}$ which satisfies the formula

$$\varphi = \exists x \forall y (\neg P(x) \wedge \neg P(y) \to P(f(y)) \wedge Q(f(y), y)) :$$

2. Construct the Herbrand model for φ from your choice above.

Question 20: Herbrand structure

Consider for this exercise first-order logic *without* equality. We fix a signature $\tau = \{P, Q, f\}$ where P is a unary predicate symbol, Q is a binary predicate symbol and f is a unary function symbol. We introduce three structures $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$.

$$\mathcal{U}^{\mathcal{A}} = \{1, 2, 3, 4\}$$

$$f^{\mathcal{A}}(x) = \begin{cases} 4 & \text{if } x = 1 \\ 1 & \text{if } x = 2 \\ 1 & \text{if } x = 3 \\ 2 & \text{if } x = 4 \end{cases}$$

$$Q^{\mathcal{A}} = \{\langle 1, 3\rangle, \langle 4, 3\rangle, \langle 2, 3\rangle\}$$

$$P^{\mathcal{A}} = \{1, 3\}$$

$$\mathcal{U}^{\mathcal{B}} = \{a, b, c, d\}$$

$$f^{\mathcal{B}}(x) = \begin{cases} d & \text{if } x = a \\ b & \text{if } x = b \\ a & \text{if } x = c \\ c & \text{if } x = d \end{cases}$$

$$Q^{\mathcal{B}} = \{\langle a, c\rangle, \langle c, d\rangle, \langle d, a\rangle\}$$

$$P^{\mathcal{B}} = \{d, c\}$$

$$\mathcal{U}^{\mathcal{C}} = \{\alpha, \beta, \gamma, \eta\}$$

$$f^{\mathcal{C}}(x) = \begin{cases} \eta & \text{if } x = \alpha \\ \alpha & \text{otherwise} \end{cases}$$

$$Q^{\mathcal{C}} = \{\langle \beta, \gamma\rangle, \langle \gamma, \eta\rangle\}$$

$$P^{\mathcal{C}} = \{\beta, \gamma, \eta\}$$

1. Pick one structure from $\mathcal{A}$, $\mathcal{B}$, or $\mathcal{C}$ which satisfies the formula

$$\varphi = \exists x \forall y \forall z (P(x) \wedge \neg Q(y, z) \to (P(y) \wedge P(z)) \wedge P(y) \to \neg P(f(y))).$$

2. Construct the Herbrand model for φ from your choice above.

Question 21: Herbrand structure

$$\varphi = \exists x \forall y \forall z ((P(x) \wedge P(f(x))) \wedge (Q(y, z) \to P(z)))$$

1. $\mathcal{A} \models \varphi$, $\mathcal{B} \not\models \varphi$, $\mathcal{C} \not\models \varphi$
2. $\psi = \forall y \forall z ((P(c) \wedge P(f(c))) \wedge (Q(y, z) \to P(z)))$
3. $c^{\mathcal{D}} = 3$
4. $f(c), f(f(f(f(f(c))))), c, f(f(f(c))), f(f(c))$
5. If we choose c, then we have $f^{\mathcal{H}}(c) = f(c)$.
6. $c \in P^{\mathcal{H}}$, $f(f(c)) \notin P^{\mathcal{H}}$
7. $(f(c), c)$, $(f^2(c), c)$, $(f^3(c), c)$, $(f^4(c), c)$, $(f^5(c), c)$

$$\varphi = \exists x \forall y (Q(f(f(x)), x) \wedge \neg P(f(y)))$$

1. $\mathcal{A} \not\models \varphi$, $\mathcal{B} \models \varphi$, $\mathcal{C} \not\models \varphi$
2. $\psi = \forall y (Q(f(f(c)), c) \wedge \neg P(f(y)))$
3. $c^{\mathcal{D}} = s$

4. $f(c), f(f(f(f(f(c))))), c, f(f(f(c))), f(f(c))$
5. If we choose c, then we have $f^{\mathcal{H}}(c) = f(c)$.
6. $c \in P^{\mathcal{H}}$, $f(c) \notin P^{\mathcal{H}}$
7. $(f^2(c), c)$, $(f^4(c), c)$, $(f^6(c), c)$, $(f^8(c), c)$, $(f^{10}(c), c)$

$$\varphi = \exists x \forall y \forall z((\neg P(x) \wedge P(f(x))) \wedge (Q(y, z) \rightarrow Q(z, y)))$$

1. $\mathcal{A} \not\models \varphi$, $\mathcal{B} \not\models \varphi$, $\mathcal{C} \models \varphi$
2. $\psi = \forall y \forall z((\neg P(c) \wedge P(f(c))) \wedge (Q(y, z) \rightarrow Q(z, y)))$
3. $c^{\mathcal{D}} = \alpha$
4. $f(c), f(f(f(f(f(c))))), c, f(f(f(c))), f(f(c))$
5. If we choose c, then we have $f^{\mathcal{H}}(c) = f(c)$.
6. $c \notin P^{\mathcal{H}}$, $f(c) \in P^{\mathcal{H}}$
7. $(c, f^2(c))$, $(f^2(c), c)$, $(f^5(c), c)$, $(c, f^5(c))$, $(f^8(c), c)$

Question 22: Theories

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be consistent theories. Prove or disprove that $\mathcal{T}_1 \cup \mathcal{T}_2$ is a theory.

Question 23: Theories

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be consistent theories. Prove or disprove that $\mathcal{T}_1 \cap \mathcal{T}_2$ is a theory.

Question 24: Theories

Let φ and ψ be two first-order formulae. Prove that

$$\varphi \equiv \psi \text{ if and only if } \mathrm{Th}(\{\varphi\}) = \mathrm{Th}(\{\psi\}).$$

Question 25: Theories

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be consistent theories. Prove or disprove that $\mathcal{T}_1 \setminus \mathcal{T}_2$ is a theory.