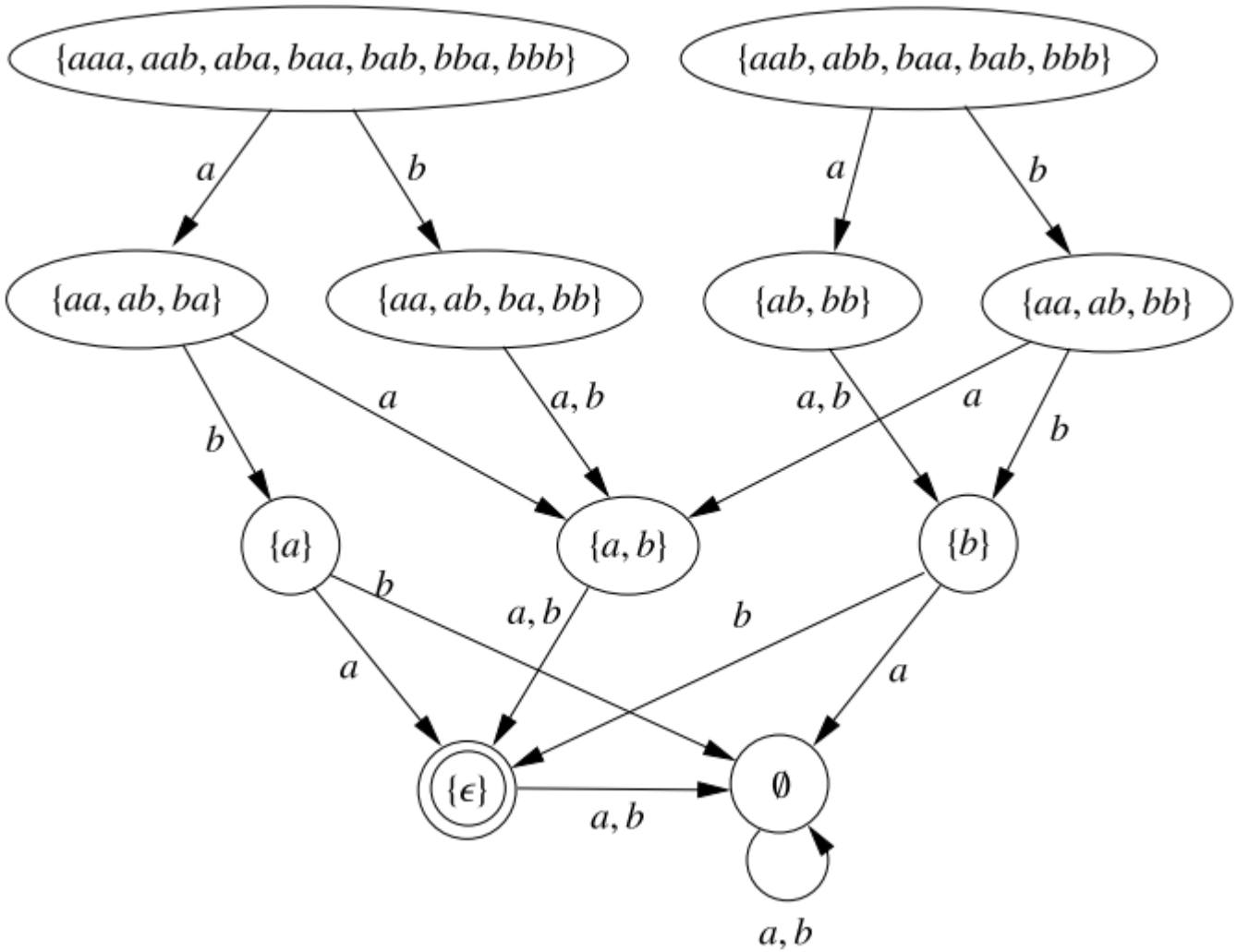
Finite Universes

Finite Universes

- When the universe is finite (e.g., the interval $[0, 2^{32} - 1]$), all objects can be encoded by words of the same length.
- A language L has length $n \geq 0$ if
  - $L = \emptyset$, or
  - every word of L has length n.
- L is a fixed-length language if it has length n for some $n \geq 0$.
- Observe:
  - Fixed-length languages contain finitely many words.
  - $\emptyset$ and $\{\varepsilon\}$ are the only two languages of length 0.
  - $\emptyset$ is a language of length n for every $n \geq 0$!

The fixed-length master automaton



The fixed-length master automaton

- The fixed-length master automaton over Σ is the tuple $M = (Q_M, \Sigma, \delta_M, F_M)$, where
  - Q_M is the set of all fixed-length languages;
  - δ_M: $Q_M \times \Sigma \to Q_M$ is given by $\delta_M(L, a) = L^a$;
  - F_M is the set $\{\{\varepsilon\}\}$ (the only final state is the language $\{\varepsilon\}$).

- Prop: The language recognized from state L of the master automaton is L.

  Proof: By induction on the length n of L.

  $n = 0$. Then either $L = \emptyset$ or $L = \{\varepsilon\}$, and result follows by inspection.

  $n > 0$. Then $\delta_M(L, a) = L^a$ for every $a \in \Sigma$, and L^a has smaller length than L. By induction hypothesis the state L^a recognizes the language L^a, and so the state L recognizes the language L.

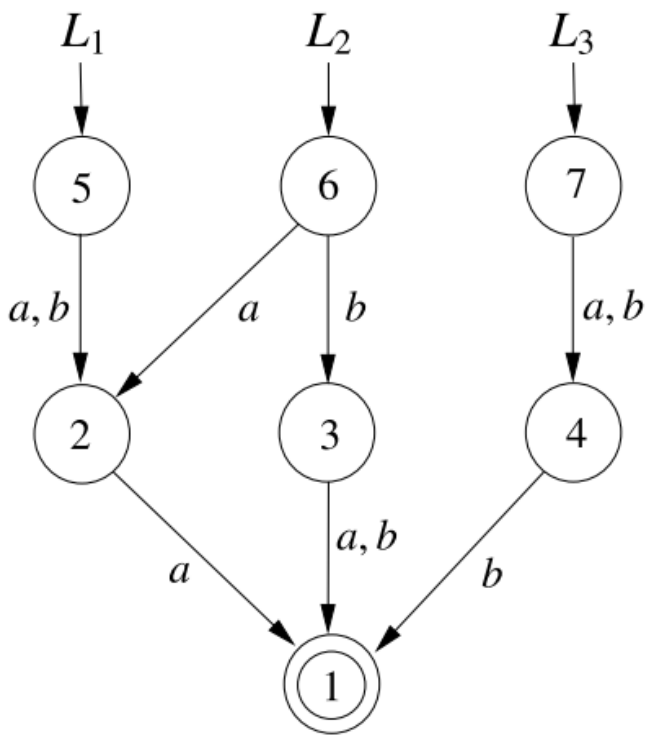
The fixed-length master automaton

- We denote the „fragment“ of the master automaton reachable from state L by A_L :
  - Initial state is L.
  - States and transitions are those reachable from L.

- Prop: A_L is the minimal DFA recognizing L.
  Proof: By definition, all states of A_L are reachable from its initial state.
  Since every state of the master automaton recognizes its „own“ language, distinct states of A_L recognize distinct languages.

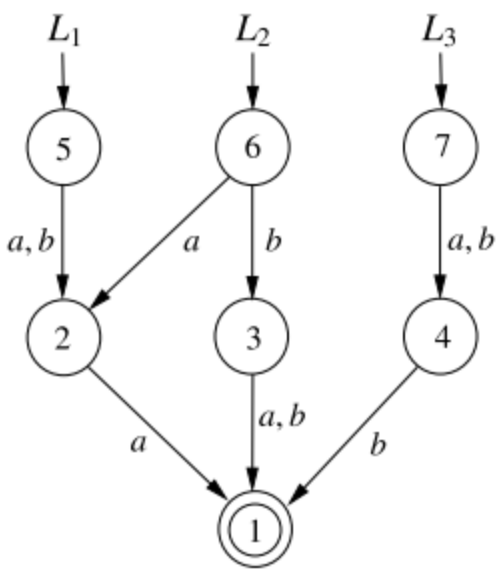
Data structure for fixed-length languages

- The structure representing the set of languages $\mathcal{L} = \{L_1, \ldots, L_m\}$ is the fragment of the master automaton containing states $L_1, \ldots, L_m$ and their descendants.
- It is a multi-DFA , i.e., a DFA with multiple initial states.



Data structure for fixed-length languages

- We represent multi-DFAs as tables of nodes .
- A node is a pair $\langle q, s \rangle$ where
  - q is a state identifier, and
  - $s = (q_1, \ldots, q_m)$ is a successor tuple.
- The table for a multi-DFA contains a node for each state but the states for $\emptyset$ and $\{\varepsilon\}$.



Ident.	a-succ	b-succ
2	1	0
3	1	1
4	0	1
5	2	2
6	2	3
7	4	4

Data structure for fixed-length languages

- The procedure $make[T](s)$
  - returns the state identifier of the node of table T having s as successor tuple, if such a node exists;
  - otherwise it adds a new node $\langle q, s \rangle$ to T, where q is a fresh identifier, and returns q.
- $make[T](s)$ assumes that T contains a node for every identifier in s.

Implementing union and intersection

- We give a recursive algorithm $inter[T](q_1, q_2)$:
  - Input: state identifiers q_1, q_2 from table T of the same length.
  - Output: identifier of the state recognizing $L(q_1) \cap L(q_2)$ in the multi-DFA for T.
  - Side-effect: if the identifier is not in T, then the algorithm adds new nodes to T, i.e., after termination the table T may have been extended.
- The algorithm follows immediately from the following properties

  (1) if $L_1 = \emptyset$ or $L_2 = \emptyset$ then $L_1 \cap L_2 = \emptyset$;

  (2) if $L_1 = \{\varepsilon\} = L_2$ then $L_1 \cap L_2 = \{\varepsilon\}$;

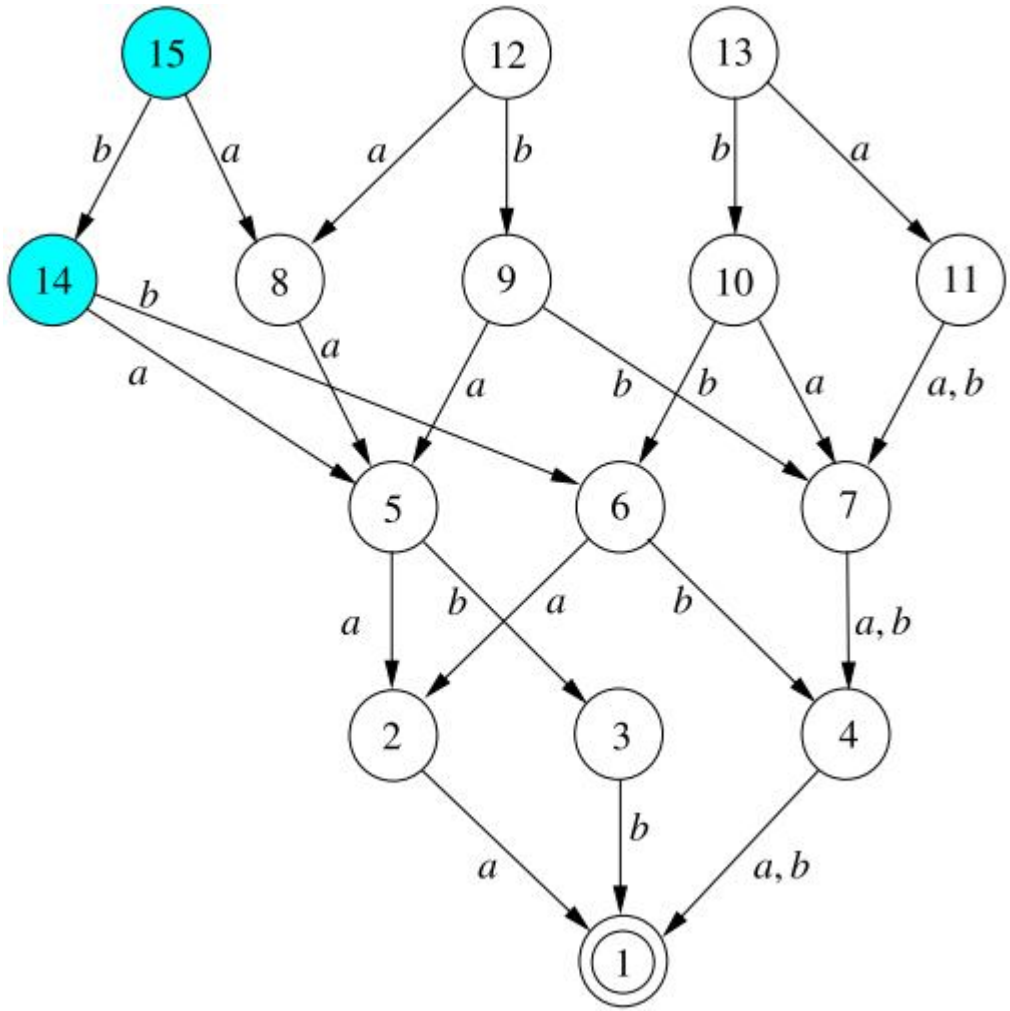
  (3) If $L_1 \neq \emptyset$ and $L_2 \neq \emptyset$, then $(L_1 \cap L_2)^a = L_1^a \cap L_2^a$ for every $a \in \Sigma$.

Implementing union and intersection

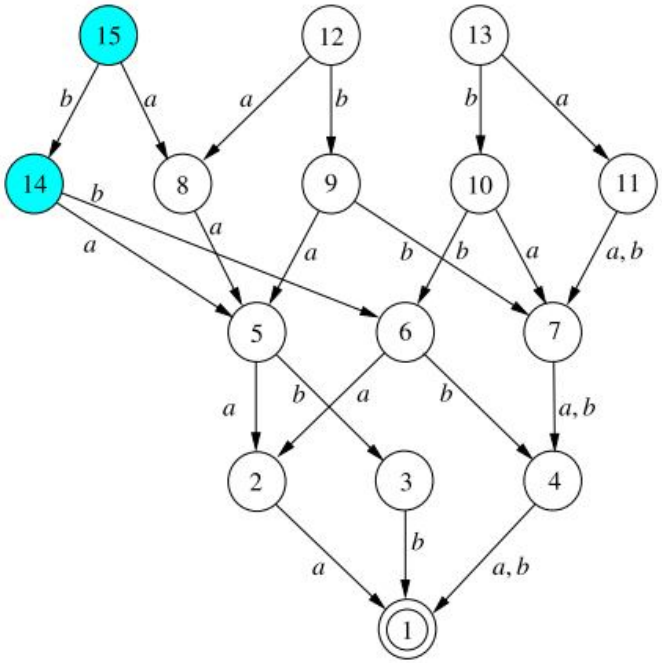
$inter(q_1, q_2)$
Input: states q_1, q_2 recognizing languages of the same length
Output: state recognizing $L(q_1) \cap L(q_2)$

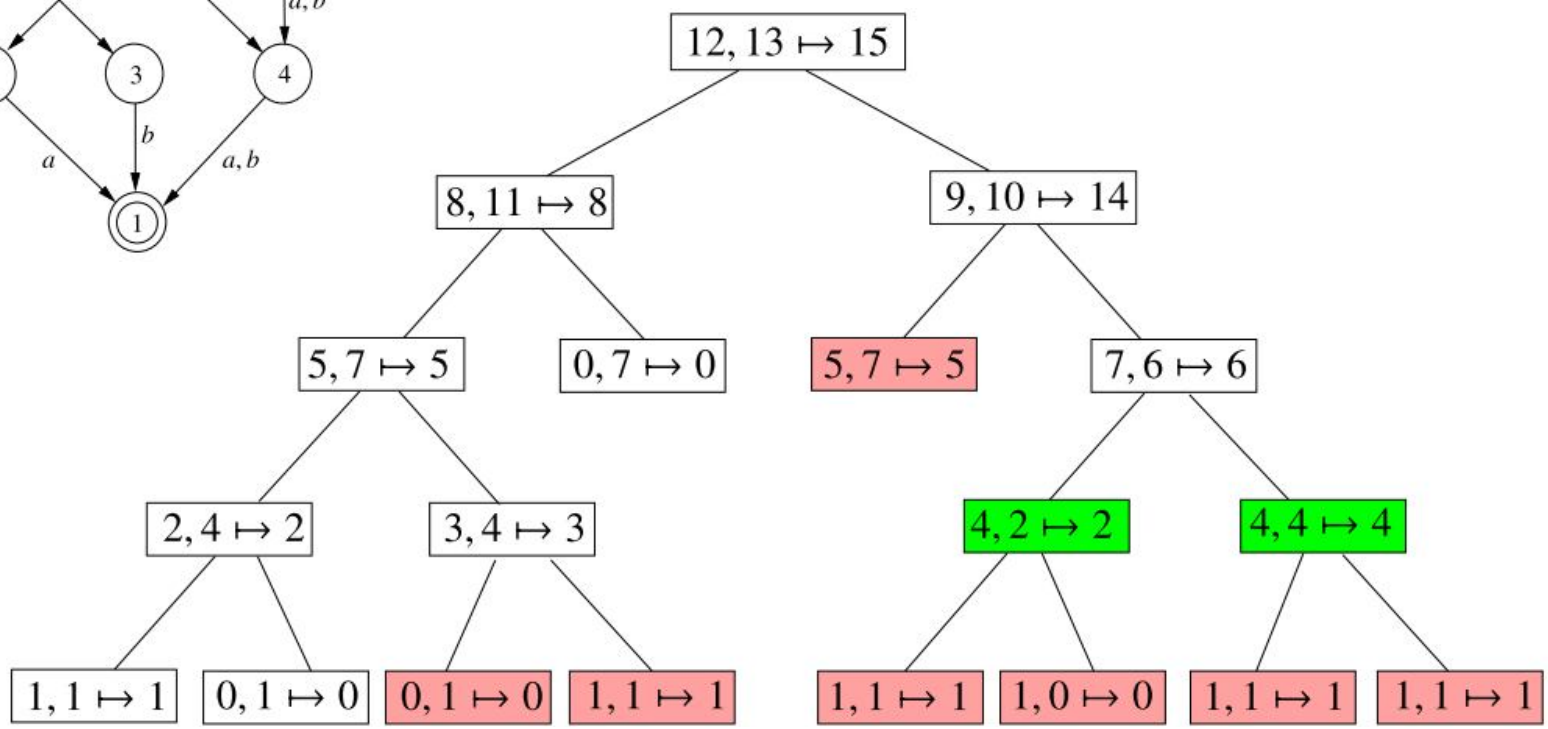
1 **if** $G(q_1, q_2)$ is not empty **then return** $G(q_1, q_2)$
2 **if** $q_1 = q_\emptyset$ **or** $q_2 = q_\emptyset$ **then return** $q_\emptyset$
3 **else if** $q_1 = q_\varepsilon$ **and** $q_2 = q_\varepsilon$ **then return** q_ϵ
4 **else** $/* \ q_1, q_2 \notin \{q_\emptyset, q_\varepsilon\} \ */$
5 **for all** $i = 1, \ldots, m$ **do** $r_i \leftarrow inter(q_1^{a_i}, q_2^{a_i})$
6 $G(q_1, q_2) \leftarrow \texttt{make}(r_1, \ldots, r_m)$
7 **return** $G(q_1, q_2)$

Implementing union and intersection



Implementing union and intersection





Implementing fixed-length complement

- If a set $X \subseteq U$ is encoded by a language L of length n, then the set $U \setminus X$ is encoded by the fixed-length complement $\Sigma^n \setminus L$, denoted by $\overline{L}^n$. This is different from $\overline{L}$!
- Since the empty language has all lengths, we have $\overline{\emptyset}^n = \Sigma^n$ for every $n \geq 0$, in particular $\overline{\emptyset}^0 = \Sigma^0 = \{\epsilon\}$,

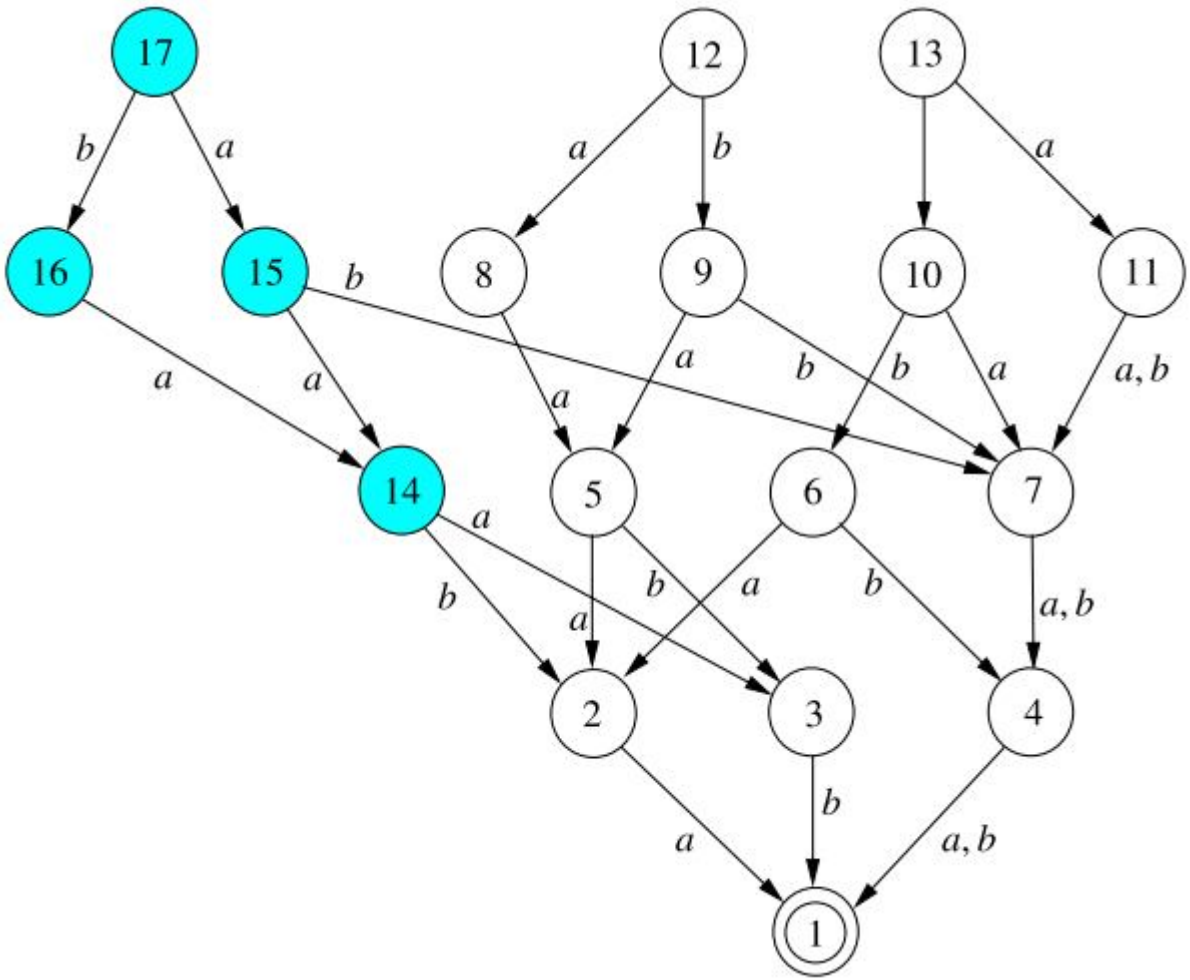
- The algorithm follows immediately from the following properties
  1. If L has length 0 and $L = \emptyset$ then $\overline{L}^0 = \{\epsilon\}$.
  2. If L has length 0 and $L = \{\varepsilon\}$ then $\overline{L}^0 = \emptyset$.
  3. If L has length $n \geq 1$, then $(\overline{L}^n)^a = \overline{L^a}^{n-1}$.

Implementing fixed-length complement

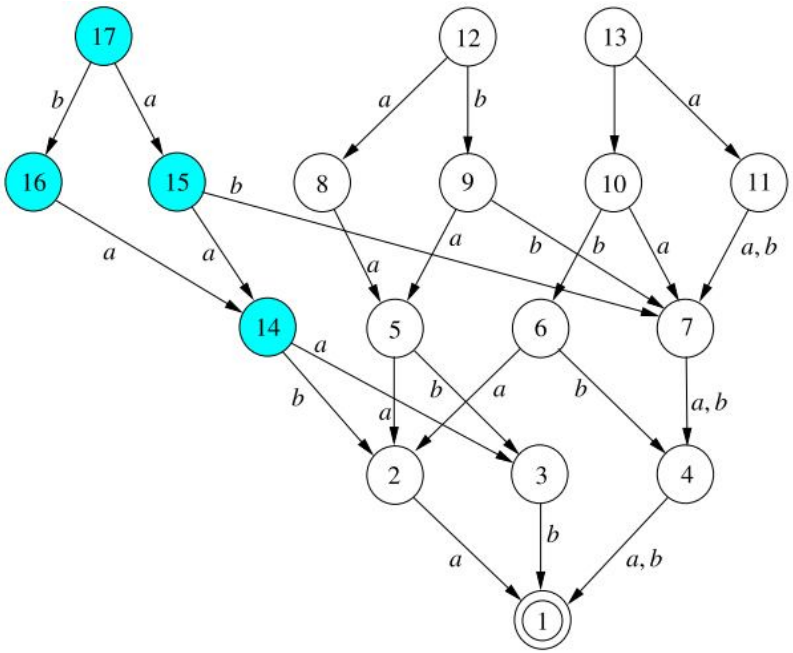
$comp(n, q)$
Input: length n, state q of length n
Output: state recognizing $\overline{L(q)}^n$

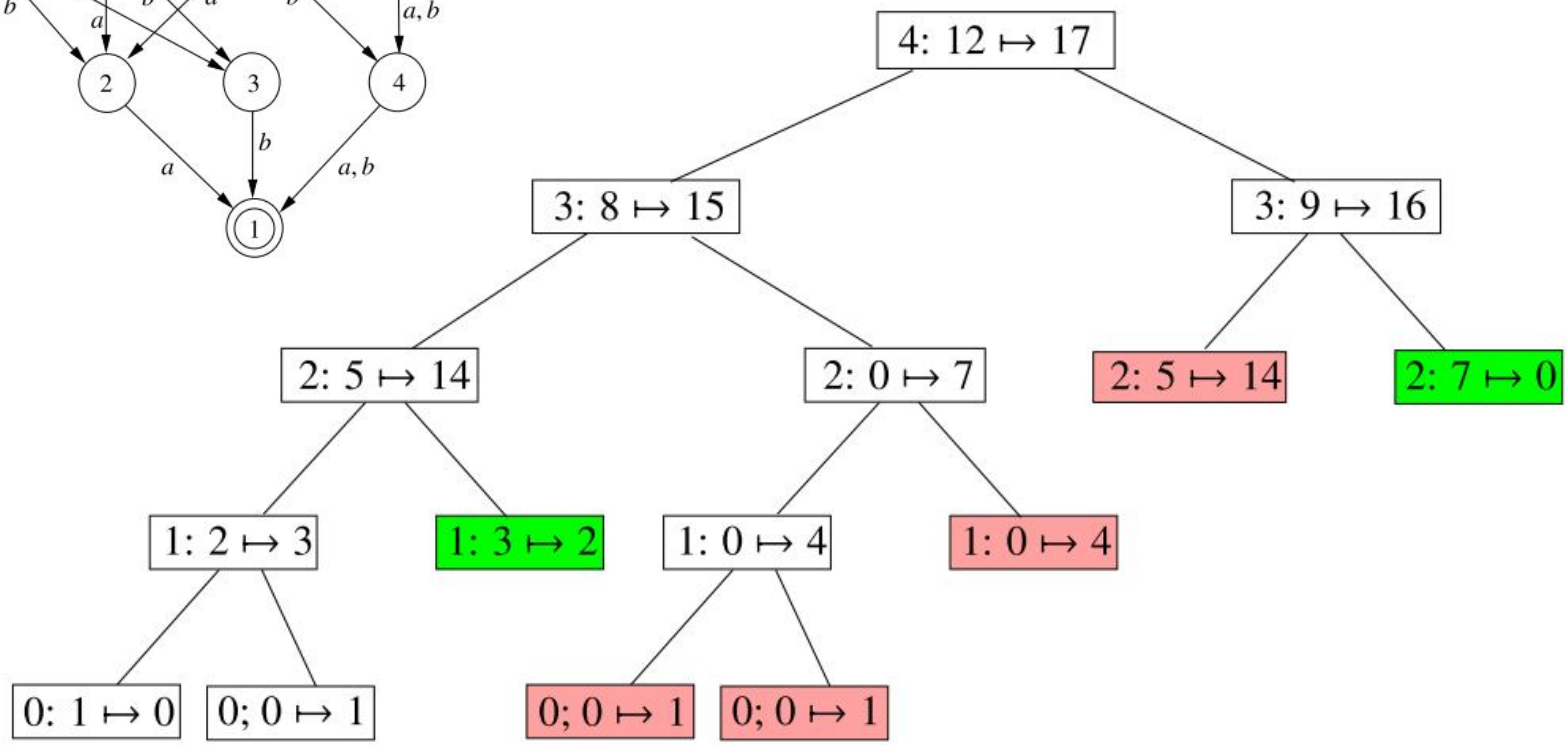
```
1  if G(n, q) is not empty then return G(n, q)
2  if n = 0 and q = q_∅ then return q_ε
3  else if n = 0 and q = q_ε then return q_∅
4  else  / * n ≥ 1 * /
5     for all i = 1, ..., m do r_i ← comp(n − 1, q^{a_i})
6     G(n, q) ← make(r_1, ..., r_m)
7     return G(n, q)
```

Implementing fixed-length complement



Implementing fixed-length complement





Implementing fixed-length universality

- A language L of length n is fixed-length universal if $L = \Sigma^n$.

- The algorithm for universality follows immediately from the following properties
  (1) If $L = \emptyset$ then L is not universal.
  (2) If $L = \{\varepsilon\}$ then L is universal.
  (3) If $\emptyset \neq L \neq \{\varepsilon\}$ then L is universal iff L^a is universal for every $a \in \Sigma$.

Implementing fixed-length universality

univ(q)
Input: state q
Output: **true** if $L(q)$ is fixed-length universal,
false otherwise

1 **if** $G(q)$ is not empty **then return** $G(q)$
2 **if** $q = q_\emptyset$ **then return false**
3 **else if** $q = q_\epsilon$ **then return true**
4 **else** $/* \; q \neq q_\emptyset$ and $q \neq q_\epsilon \; */$
5 $G(q) \leftarrow$ **and**$(univ(q^{a_1}), \ldots, univ(q^{a_m}))$
6 **return** $G(q)$

Implementing fixed-length equality

- If two languages L_1, L_2 of the same length are represented by nodes q_1, q_2 of the same table then we have $L_1 = L_2$ iff $q_1 = q_2$, and so equality can be checked in constant time.
- If the languages are represented by nodes from different tables, then equality amounts to isomorphism of the DFAs rooted at the nodes.

$eq2(q_1, q_2)$
Input: states q_1, q_2 of different tables
Output: **true** if $L(q_1) = L(q_2)$, **false** otherwise

1 **if** $G(q_1, q_2)$ is not empty **then return** $G(q_1, q_2)$
2 **if** $q_1 = q_{\emptyset 1}$ and $q_2 = q_{\emptyset 2}$ **then** $G(q_1, q_2) \leftarrow$ `true`
3 **else if** $q_1 = q_{\emptyset 1}$ and $q_2 \neq q_{\emptyset 2}$ **then** $G(q_1, q_2) \leftarrow$ `false`
4 **else if** $q_1 \neq q_{\emptyset 1}$ and $q_2 = q_{\emptyset 2}$ **then** $G(q_1, q_2) \leftarrow$ `false`
5 **else** $/* q_1 \neq q_{\emptyset 1}$ and $q_2 \neq q_{\emptyset 2} */$
6 $G(q_1, q_2) \leftarrow$ **and**$(\mathrm{eq}(q_1^{a_1}, q_2^{a_1}), \ldots, \mathrm{eq}(q_1^{a_m}, q_2^{a_m}))$
7 **return** $G(q_1, q_2)$

NFAs as starting point

- Given: Acyclic NFA A accepting a fixed-length language.

  Goal: Simultaneously determinize and minimize A
- Each state of A accepts a fixed-length language.
- We give an algorithm *state*(S):
  - Input: a subset S of states of A accepting languages of the same length.
  - Output: the state of the master automaton accepting $\mathrm{U}_{q \in S} L(q)$.
- Goal is achieved by calling *state*(Q_0)

NFAs as starting point

- The algorithm follows from the following observations:

1) If $S = \emptyset$ then $L(S) = \emptyset$.
2) If $S \cap F \neq \emptyset$ then $L(S) = \{\epsilon\}$.
3) If $S \neq \emptyset$ and $S \cap F = \emptyset$ then $L(S) = \bigcup_{i=1}^{n} a_i \cdot L(S_i)$, where $L(S_i) = \delta(S, a_i)$.

- This leads directly to a recursive algorithm:

NFAs as starting point

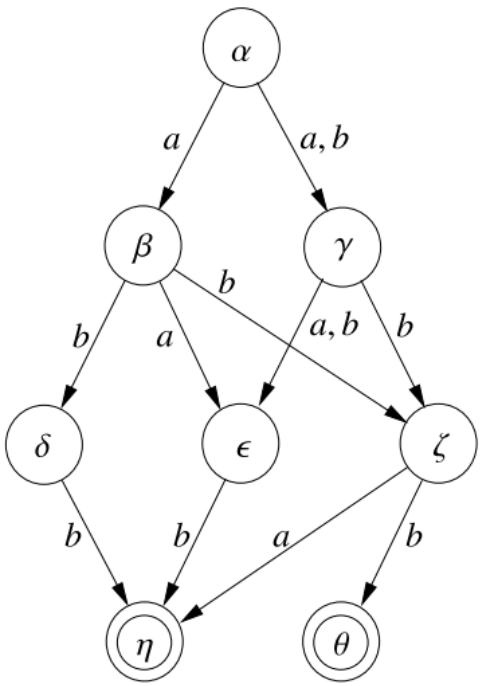
det&min(A)
Input: NFA $A = (Q, \Sigma, \delta, Q_0, F)$
Output: master state recognizing $L(A)$

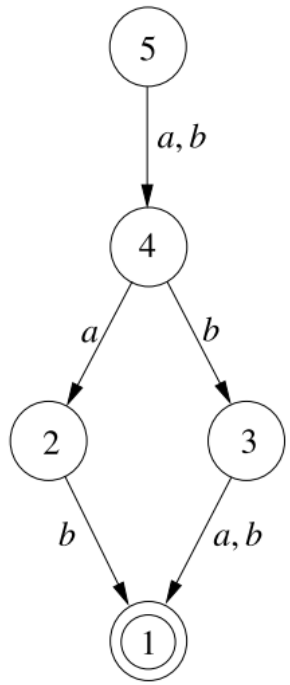
```
1    return state(Q_0)
```

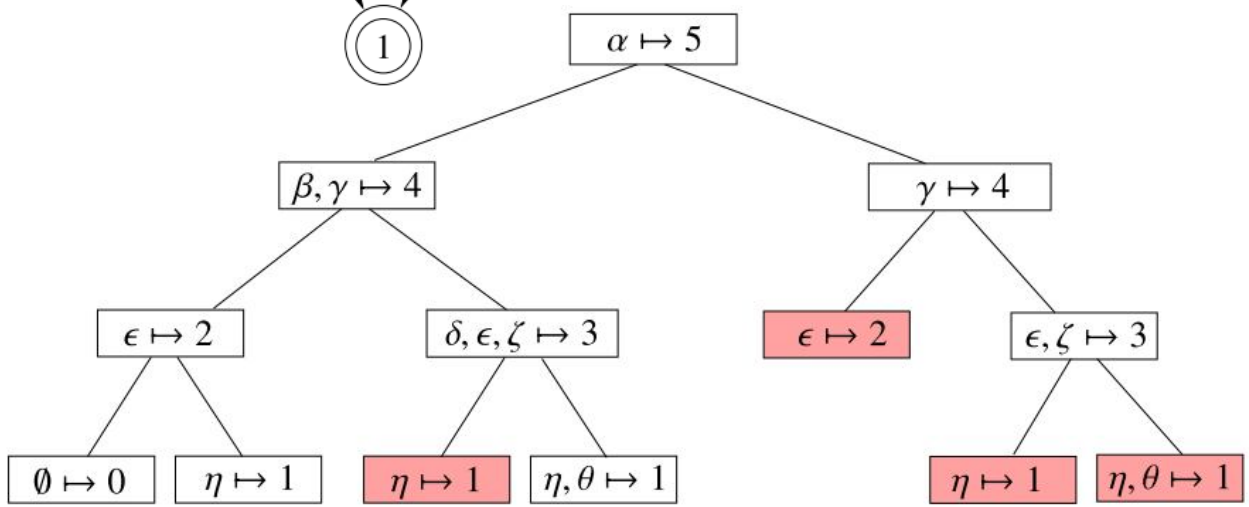
state(S)
Input: set $S \subseteq Q$ recognizing languages of the same length
Output: state recognizing $L(S)$

```
1    if G(S) is not empty then return G(S)
2    else if S = ∅ then return q_∅
3    else if S ∩ F ≠ ∅ then return q_ε
4    else  /* S ≠ ∅ and S ∩ F = ∅ */
5        for all i = 1, ..., m do S_i ← δ(S, a_i)
6        G(S) ← make(state(S_1), ..., state(S_m));
7        return G(S)
```

NFAs as starting point







Implementing operations on relations

- Assumptions:
  - Objects are encoded as words of Σ^n (one word for each object)
  - Pairs of objects are encoded as words of $(\Sigma \times \Sigma)^n$.
    Recall: $\Sigma^n \times \Sigma^n$ and $(\Sigma \times \Sigma)^n$ are isomorphic.
  - Observe: objects and pairs of objects are both encoded as words of length n, but over different alphabets.
- Notation: Given $R \subseteq \Sigma^n \times \Sigma^n$, we denote

  $$R^{[a,b]} = \{ (w_1, w_2) \in \Sigma^{n-1} \times \Sigma^{n-1} \mid (aw_1, bw_2) \in R \}.$$
- Master transducer: Master automaton over the alphabet $\Sigma \times \Sigma$.

Implementing fixed-length join

- The algorithm follows from:

  1) $\emptyset \circ R = R \circ \emptyset = \emptyset$

  2) $\{(\epsilon, \epsilon)\} \circ \{(\epsilon, \epsilon)\} = \{(\epsilon, \epsilon)\}$

  3) If R_1, R_2 have length at least 1, then

$$R_1 \circ R_2 = \bigcup_{a,b,c \in \Sigma} [a, b] \cdot \left(R_1^{[a,c]} \circ R_2^{[c,b]} \right)$$

Implementing fixed-length join

$join(r_1, r_2)$

Input: states r_1, r_2 of transducer table

Output: state recognizing $L(r_1) \circ L(r_2)$

1 **if** $G(r_1, r_2)$ is not empty **then return** $G(r_1, r_2)$

2 **if** $r_1 = q_\emptyset$ **or** $r_2 = q_\emptyset$ **then return** $q_\emptyset$

3 **else if** $r_1 = q_\epsilon$ **and** $r_2 = q_\epsilon$ **then return** q_ϵ

4 **else** $/* \; q_\emptyset \neq r_1 \neq q_\epsilon$ and $q_\emptyset \neq r_2 \neq q_\epsilon \; */$

5 **for all** $(a_i, a_j) \in \Sigma \times \Sigma$ **do**

6 $r_{i,j} \leftarrow union\left(join\left(r_1^{[a_i,a_1]}, r_2^{[a_1,a_j]}\right), \ldots, join\left(r_1^{[a_i,a_m]}, r_2^{[a_m,a_j]}\right)\right)$

7 $G(r_1, r_2) = make(r_{1,1}, \ldots, \ldots, r_{m,m})$

8 **return** $G(r_1, r_2)$

Implementing fixed-length pre and post

- The algorithm for *pre* (*post* is analogous) follows from:

  1) If $R = \emptyset$ or $L = \emptyset$ then $pre_{R(L)} = \emptyset$

  2) If $R = \{[\epsilon, \epsilon]\}$ and $L = \{\epsilon\}$ then $pre_{R(L)} = \{\epsilon\}$

  3) If $\emptyset \neq R \neq \{[\epsilon, \epsilon]\}$ and $\emptyset \neq L \neq \{\epsilon\}$ then

$$pre_R(L) = \bigcup_{a,b \in \Sigma} a \cdot pre_{R^{[a,b]}}(L^b)$$

Proof of 3):

$$\begin{aligned}
& aw_1 \in pre_R(L) \\
\Leftrightarrow \quad & \exists bw_2 \in L\colon [aw_1, bw_2] \in R \\
\Leftrightarrow \quad & \exists b \in \Sigma\ \exists w_2 \in L^b\colon [w_1, w_2] \in R^{[a,b]} \\
\Leftrightarrow \quad & \exists b \in \Sigma\colon w_1 \in pre_{R^{[a,b]}}(L^b) \\
\Leftrightarrow \quad & aw_1 \in \bigcup_{b \in \Sigma} a \cdot pre_{R^{[a,b]}}(L^b)
\end{aligned}$$

Implementing fixed-length pre and post

$pre(r, q)$

Input: state r of transducer table, state q of automaton table

Output: state recognizing $pre_{L(r)}(L(q))$

1 **if** $G(r, q)$ is not empty **then return** $G(r, q)$

2 **if** $r = r_\emptyset$ **or** $q = q_\emptyset$ **then return** $q_\emptyset$

3 **else if** $r = r_\epsilon$ **and** $q = q_\epsilon$ **then return** q_ϵ

4 **else**

5 **for all** $a_i \in \Sigma$ **do**

6 $q'_i \leftarrow union\left(pre\left(r^{[a_i,a_1]}, q^{a_1}\right), \ldots, pre\left(r^{[a_i,a_m]}, q^{a_m}\right)\right)$

7 $G(q, r) \leftarrow make(q'_1, \ldots, q'_m)$

8 **return** $G(q, r)$

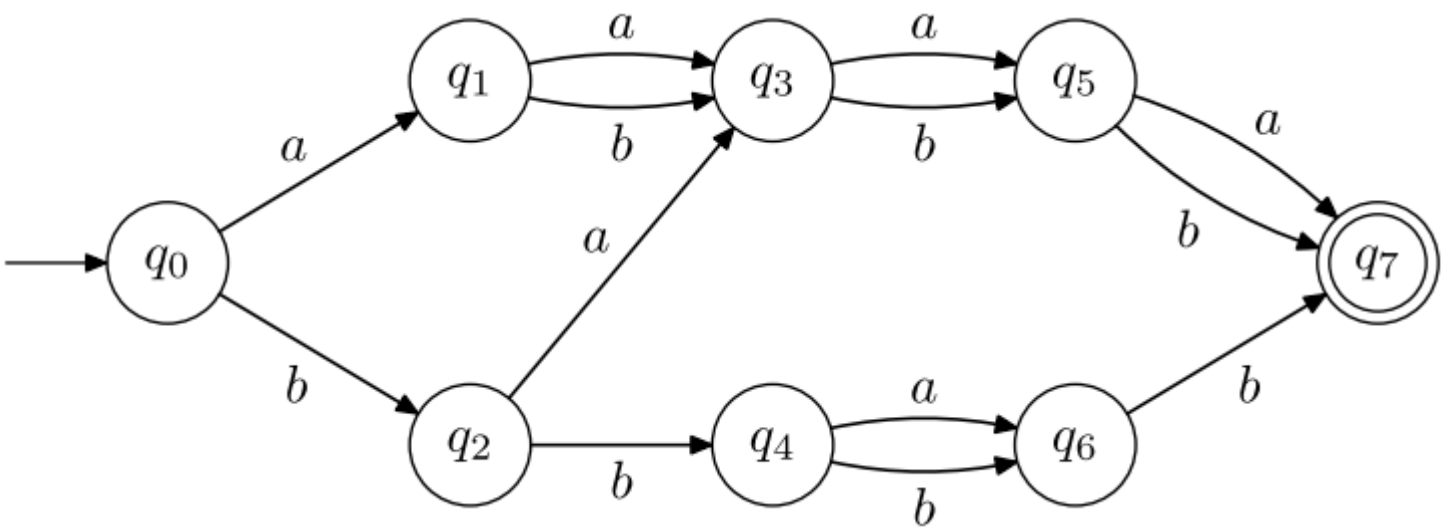
Implementing projection

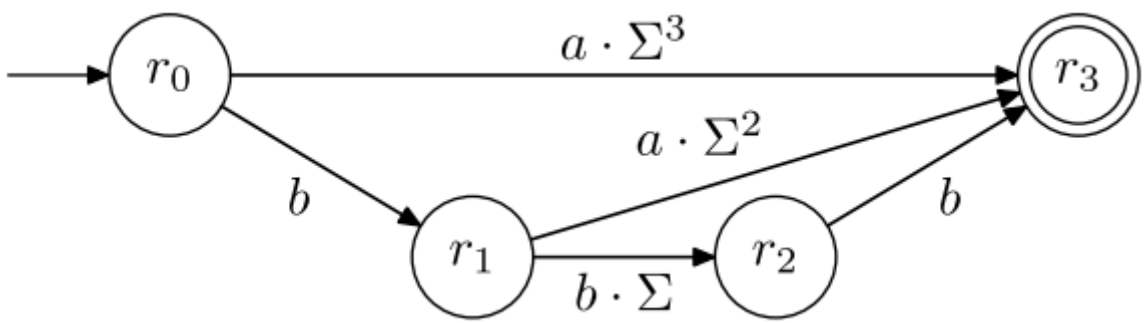
- We reduce projection to *pre*.
- The projection of a language $R \subseteq \Sigma^n \times \Sigma^n$ onto the first component is the language $pre_R(\Sigma^n)$.
- Specializing the algorithm for *pre* we obtain:

$pro_1(r)$
Input: state r of transducer table
Output: state recognizing $proj_1(L(r))$

1 **if** $G(r)$ is not empty **then return** $G(r)$
2 **if** $r = r_\emptyset$ **then return** $q_\emptyset$
3 **else if** $r = r_\epsilon$ **then return** q_ϵ
4 **else**
5 **for all** $a_i \in \Sigma$ **do**
6 $q'_i \leftarrow union\left(pro_1\left(r^{[a_i,a_1]}\right), \ldots, pro_1\left(r^{[a_i,a_m]}\right)\right)$
7 $G(r) \leftarrow make(q'_1, \ldots, q'_m)$
8 **return** $G(r)$

Decision Diagrams (DDs)



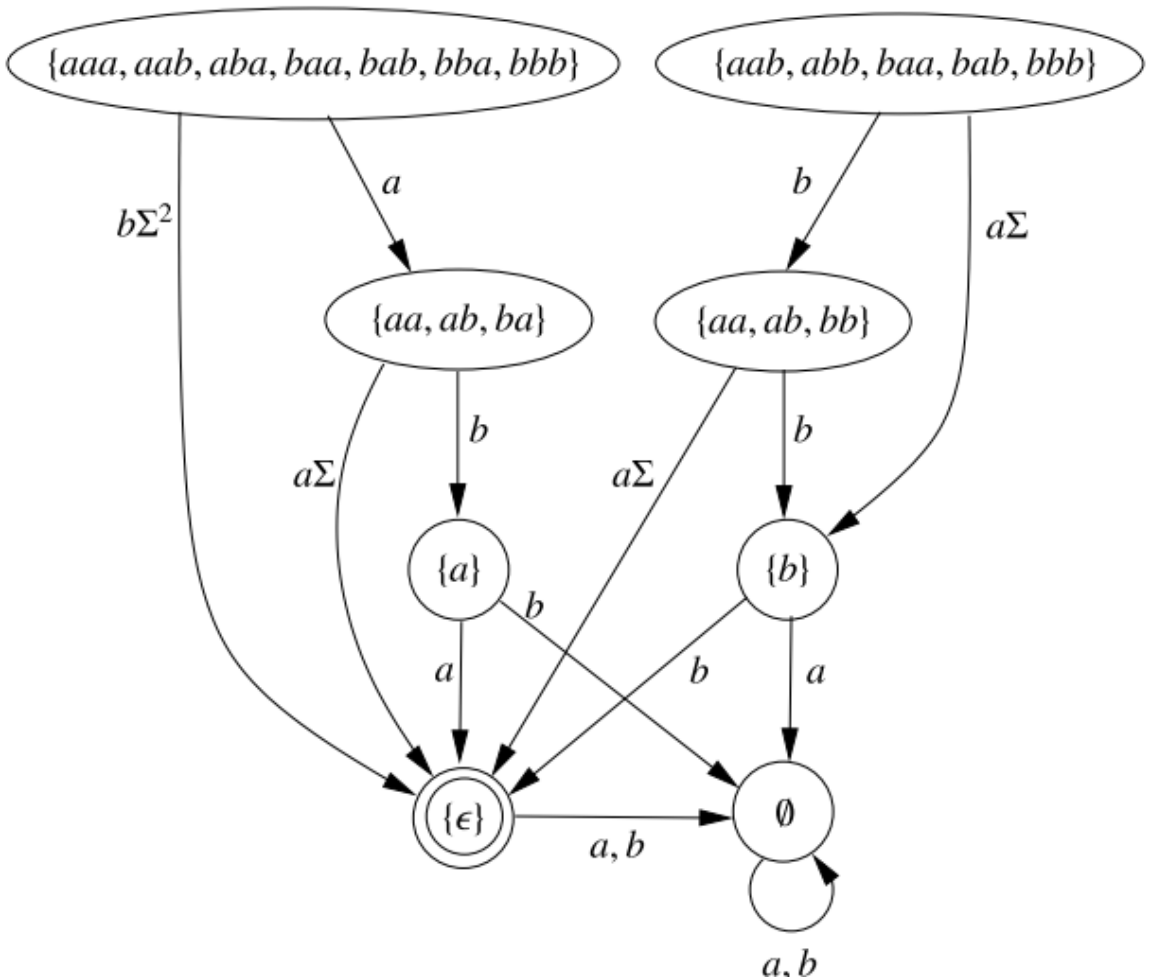


Decision Diagrams (DDs)

- A decision diagram is an automaton
  - whose transitions are labeled by regular expressions of the form $a\Sigma^n$, $n \geq 0$, and
  - satisfies the following determinacy condition for every state q and letter a: there is exactly one $k \geq 0$ such that $\delta(q, a\Sigma^k) \neq \emptyset$, and for this k there is a state q' such that $\delta(q, a\Sigma^k) = \{q'\}$.
- Observe: Every DFA is a DD.
- A fixed-length language L is a kernel if $L = \emptyset$, $L = \{\epsilon\}$, or there are $a, b \in \Sigma$ such that $L^a \neq L^b$.
- The kernel $\langle L \rangle$ of a fixed-length language L is the unique kernel satisfying $L = \Sigma^k \langle L \rangle$ for some $k \geq 0$. Observe: k and $\langle L \rangle$ uniquely determine L for every $L \neq \emptyset$.

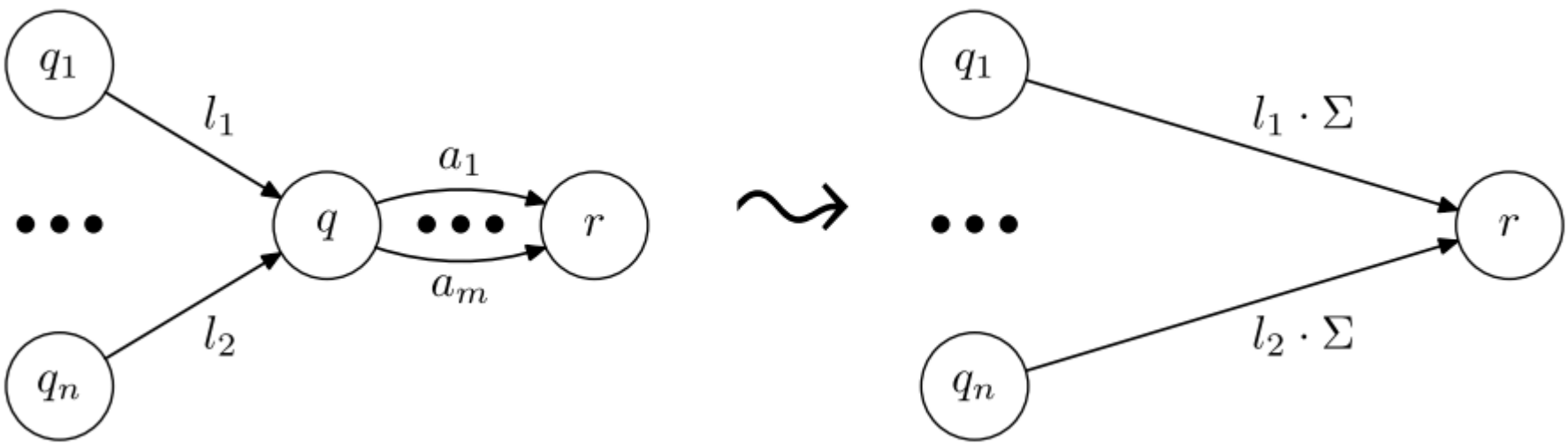
The fixed-length master decision diagram

- All kernels as states, $\{\epsilon\}$ as final state, transitions $(K, a\Sigma^k, \langle K^a \rangle)$



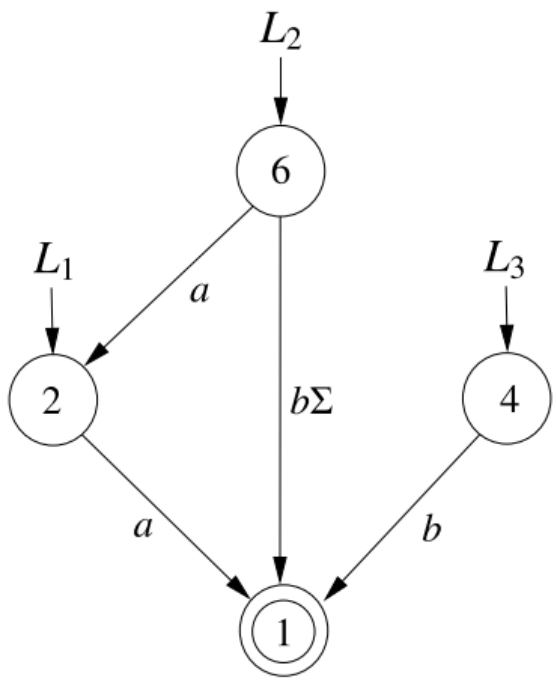
Reduction rule

- Proposition: The unique minimal DD for a kernel is the fragment of the fixed-length master DD rooted at the kernel (modulo labels of transitions leaving the states $\emptyset$ and $\{\epsilon\}$).
- Proposition: The minimal DD for a kernel is obtained from its minimal DFA by exhaustively applying the following „reduction rule“:



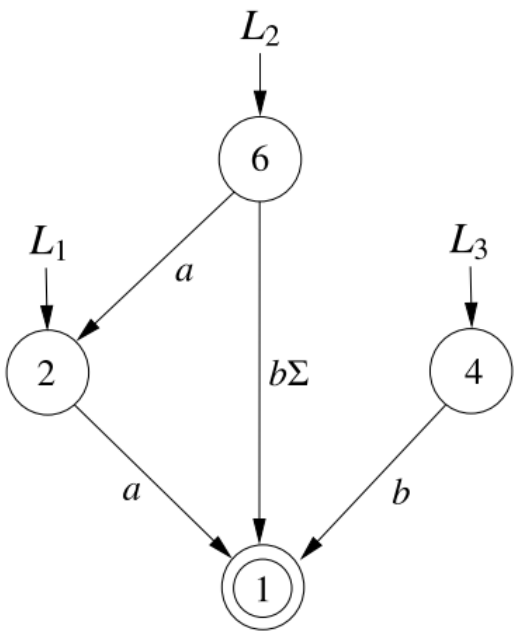
Data structure for kernels

- The structure representing the set of kernels $\mathcal{L} = \{L_1, \ldots, L_m\}$ is the fragment of the master DD containing states $L_1, \ldots, L_m$ and their descendants.
- It is a multi-DD , i.e., a DD with multiple initial states.



Data structure for kernels

- We represent multi-DDs as tables of kernodes .
- A kernode is a triple $\langle q, l, s \rangle$ where
  - q is a state identifier,
  - l is a length, and
  - $s = (q_1, \ldots, q_m)$ is a successor tuple.
- The table for a multi-DD contains a node for each state but the states for $\emptyset$ and ϵ.



Ident.	Length	a-succ	b-succ
2	1	1	0
4	1	0	1
6	2	2	1

Implementing intersection

- Given kernels K_1, K_2 of languages L_1, L_2, we wish to compute $K_1 \sqcap K_2 := \langle L_1 \cap L_2 \rangle$.

- Assume $L_1 = \Sigma^{l_1} K_1$ and $L_2 = \Sigma^{l_2} K_2$. We have
  1. If $K_1 = \emptyset$ or $K_2 = \emptyset$ then $K_1 \sqcap K_2 = \emptyset$.
  2. If $K_1 \neq \emptyset \neq K_2$ then

$$K_1 \sqcap K_2 = \begin{cases} \langle K_1 \cap \Sigma^{l_2 - l_1} K_2 \rangle & \text{if } l_1 < l_2 \\ \langle \Sigma^{l_1 - l_2} K_1 \cap K_2 \rangle & \text{if } l_2 < l_1 \\ \langle K_1 \cap K_2 \rangle & \text{if } l_1 = l_2 \end{cases}$$

  3. If $l_1 < l_2$ then $\langle (K_1 \cap \Sigma^{l_2 - l_1} K_2)^a \rangle = \langle K_1^a \rangle \sqcap K_2$
  4. If $l_2 < l_1$ then $\langle (\Sigma^{l_2 - l_1} K_1 \cap K_2)^a \rangle = K_1 \sqcap \langle K_2^a \rangle$
  5. If $l_1 = l_2$ then $\langle (K_1 \cap K_2)^a \rangle = \langle K_1^a \rangle \sqcap \langle K_2^a \rangle$

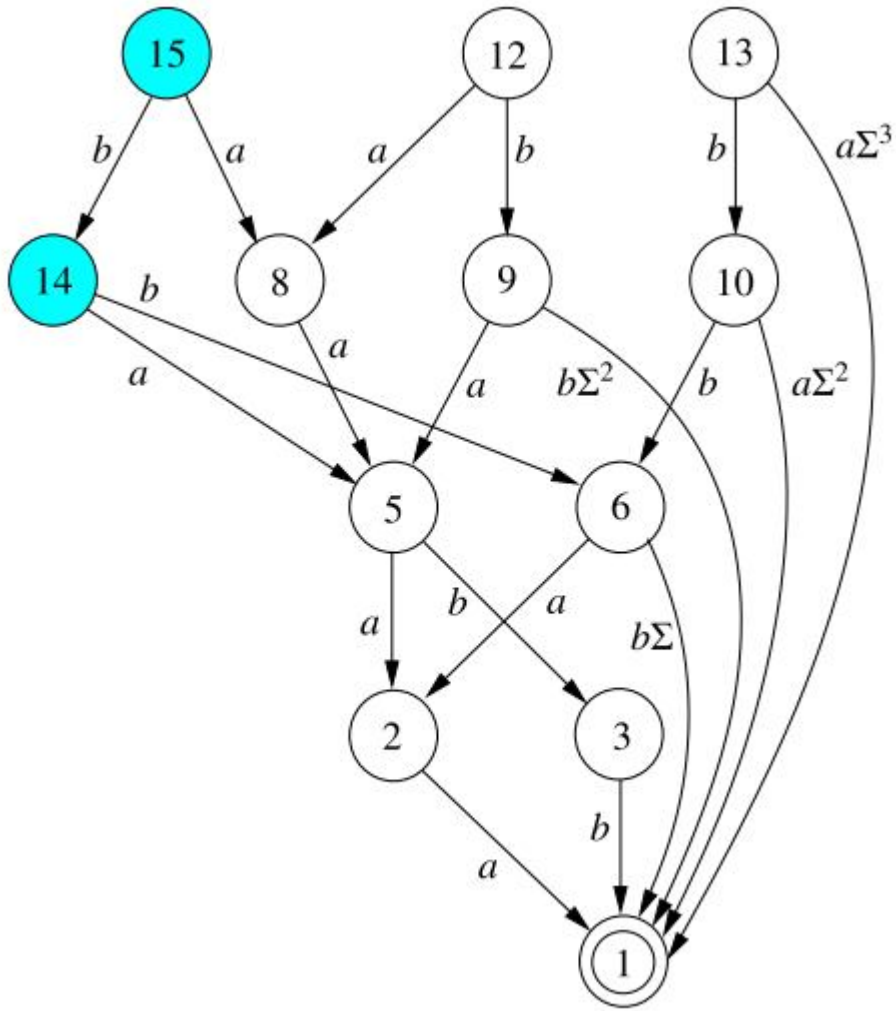
- 3.-5. lead to a recursive algorithm

Implementing intersection

kinter(q_1, q_2)
Input: states q_1, q_2 recognizing $\langle L_1 \rangle, \langle L_2 \rangle$
Output: state recognizing $\langle L_1 \cap L_2 \rangle$

```
1   if G(q1, q2) is not empty  then return G(q1, q2)
2   if q1 = q∅ or q2 = q∅ then return q∅
3   if q1 ≠ q∅ and q2 ≠ q∅ then
4      if l1 < l2  /* lengths of the kernodes for q1, q2 */ then
5         for all i = 1, ..., m do ri ← kinter(q1, q2^ai)
6         G(q1, q2) ← kmake(l2, r1, ..., rm)
7      else if l1    l2 then
8         for all i = 1, ..., m do ri ← kinter(q1^ai, q2)
9         G(q1, q2) ← kmake(l1, r1, ..., rm)
10     else /* l1 = l2 */
11        for all i = 1, ..., m do ri ← kinter(q1^ai, q2^ai)
12        G(q1, q2) ← kmake(l1, r1, ..., rm)
13  return G(q1, q2)
```

Implementing intersection



Implementing intersection