Pattern Matching

Pattern Matching

- Given
  - a word t (the text) of length n, and
  - a regular expression p (the pattern) of length m

  determine
  - the smallest number k' such that some $[k, k']$-factor of w belongs to $L(p)$.

NFA-based solution

PatternMatchingNFA(t, p)

Input: text $t = a_1 \dots a_n \in \Sigma^+$, pattern $p \in \Sigma^*$

Output: the first occurrence of p in t, or $\perp$ if no such occurrence exists.

```
1  A ← RegtoNFA(Σ* p)
2  S ← Q_0
3  for all k = 0 to n − 1 do
4      if S ∩ F ≠ ∅ then return k
5      S ← δ(S, a_{k+1})
6  return ⊥
```

- Line 1 takes $O(m^3)$ time ($O(m^2)$ for fixed alphabet) , output has $O(m)$ states
- Loop is executed at most n times
- One iteration takes $O(s^2)$ time , where s is the number of states of A
- Since $s = O(m)$, the total runtime is $O(m^3 + nm^2)$, and $O(nm^2)$ for $m \leq n$.

DFA-based solution

PatternMatchingDFA(t, p)

Input: text $t = a_1 \ldots a_n \in \Sigma^+$, pattern p

Output: the first occurrence of p in t, or $\perp$ if no such occurrence exists.

1 $A \leftarrow \mathit{NFAtoDFA}(\mathit{RegtoNFA}(\Sigma^* p))$

2 $q \leftarrow q_0$

3 **for all** $k = 0$ to $n - 1$ **do**

4   **if** $q \in F$ **then return** k

5     $q \leftarrow \delta(q, a_{k+1})$

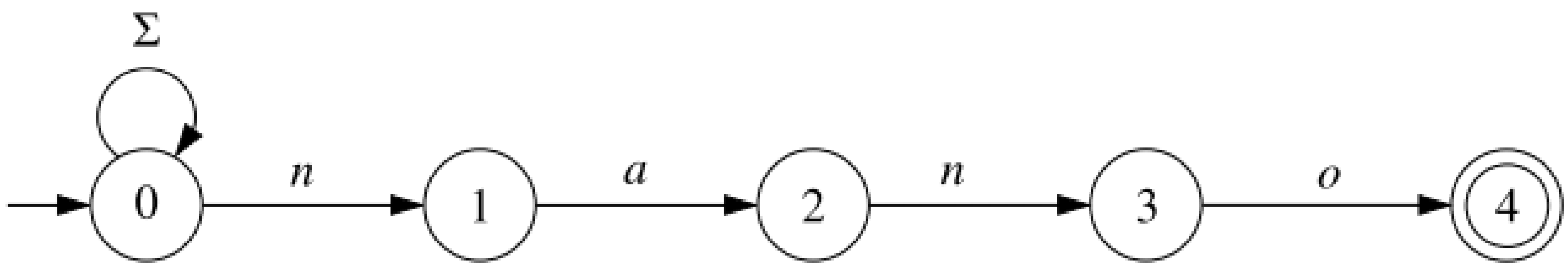
6 **return** $\perp$

- Line 1 takes $2^{O(m)}$ time
- Loop is executed at most n times
- One iteration takes constant time
- Total runtime is $O(n) + 2^{O(m)}$

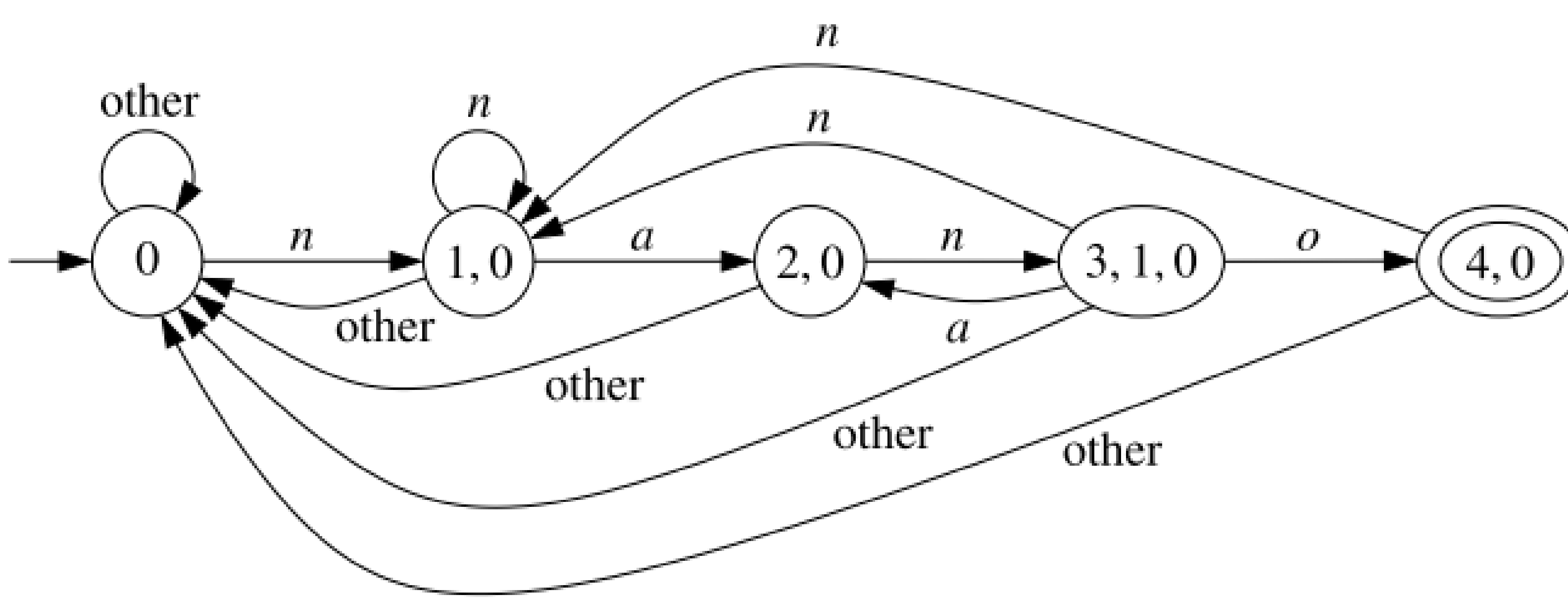
The word case

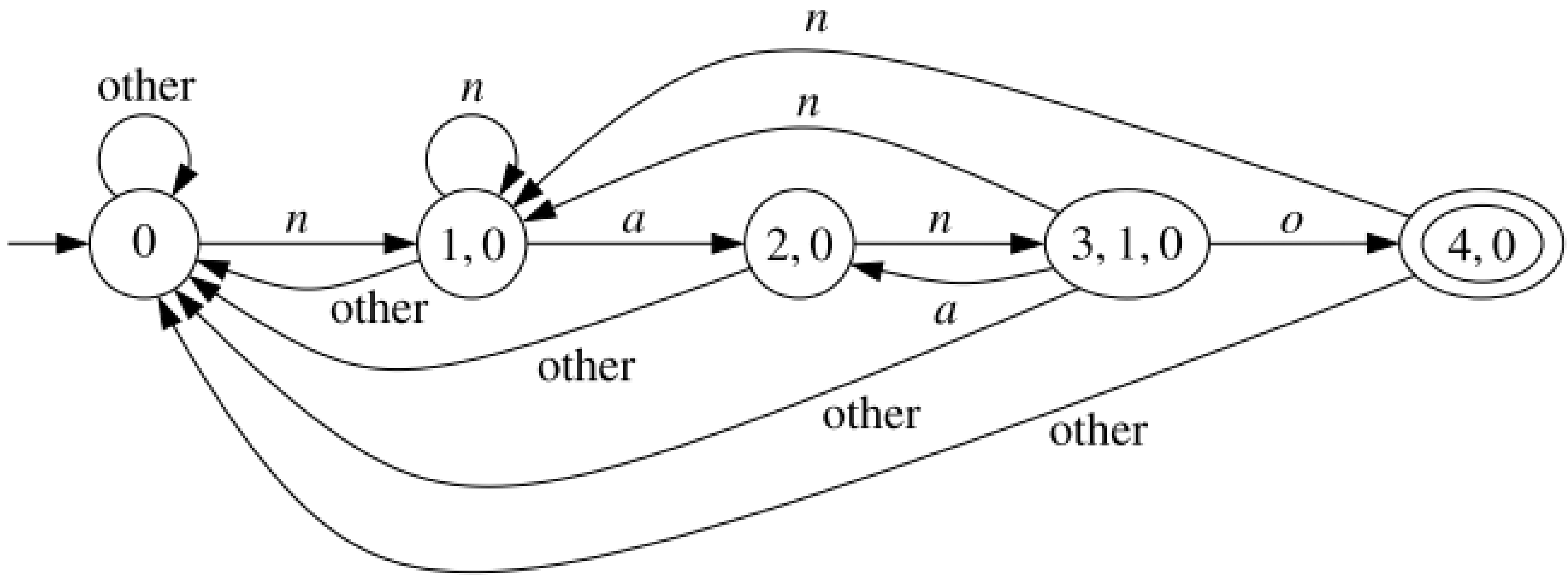
- The pattern $p = b_1 b_2 \ldots b_m$ is a word of length m
- Naive algorithm: move a window of size m along the word one letter at a time, and compare with *p* after each step. Runtime: $O(nm)$ for any alphabet of size $0 \leq |\Sigma| \leq n$.
- We give an algorithm with $O(n + m)$ runtime for any alphabet of size $0 \leq |\Sigma| \leq n$.
- First we explore in detail the shape of the DFA for $\Sigma^* p$.

Obvious NFA for $\Sigma^* p$ and $p = nano$

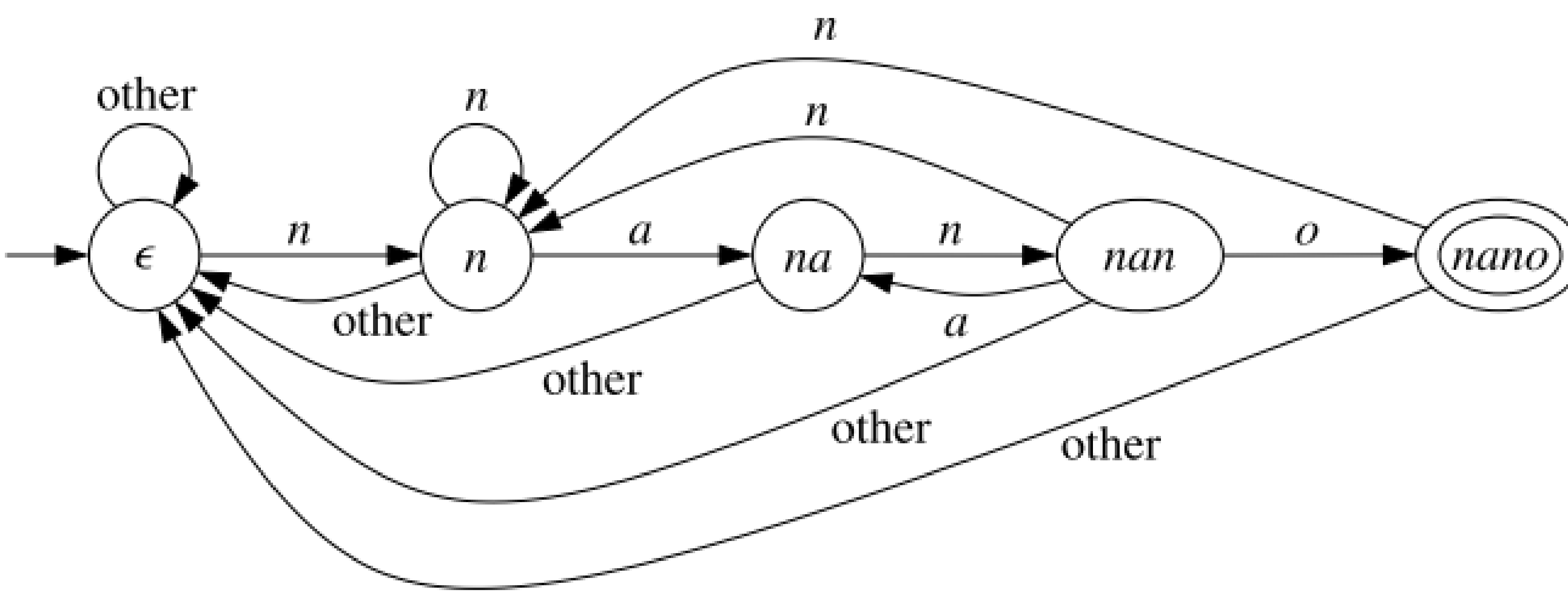


Result of applying NFAtoDFA:





other
n
n
n
0
n
1, 0
a
2, 0
n
3, 1, 0
o
4, 0
other
a
other
other
other



other
n
n
n
ϵ
n
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nan
o
nano
other
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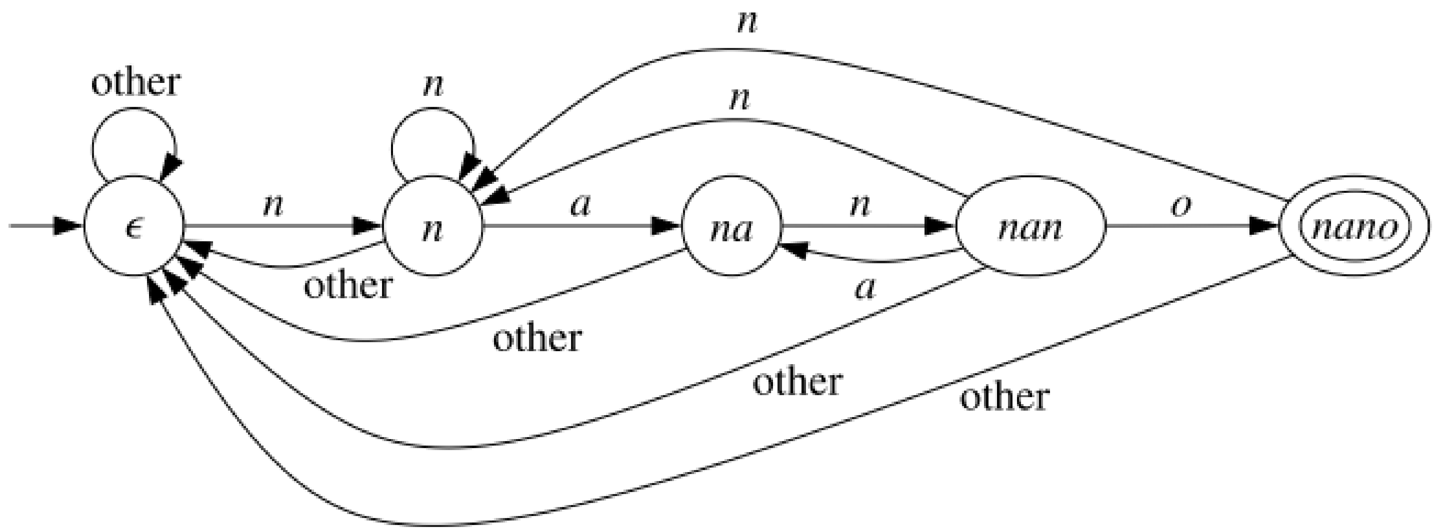
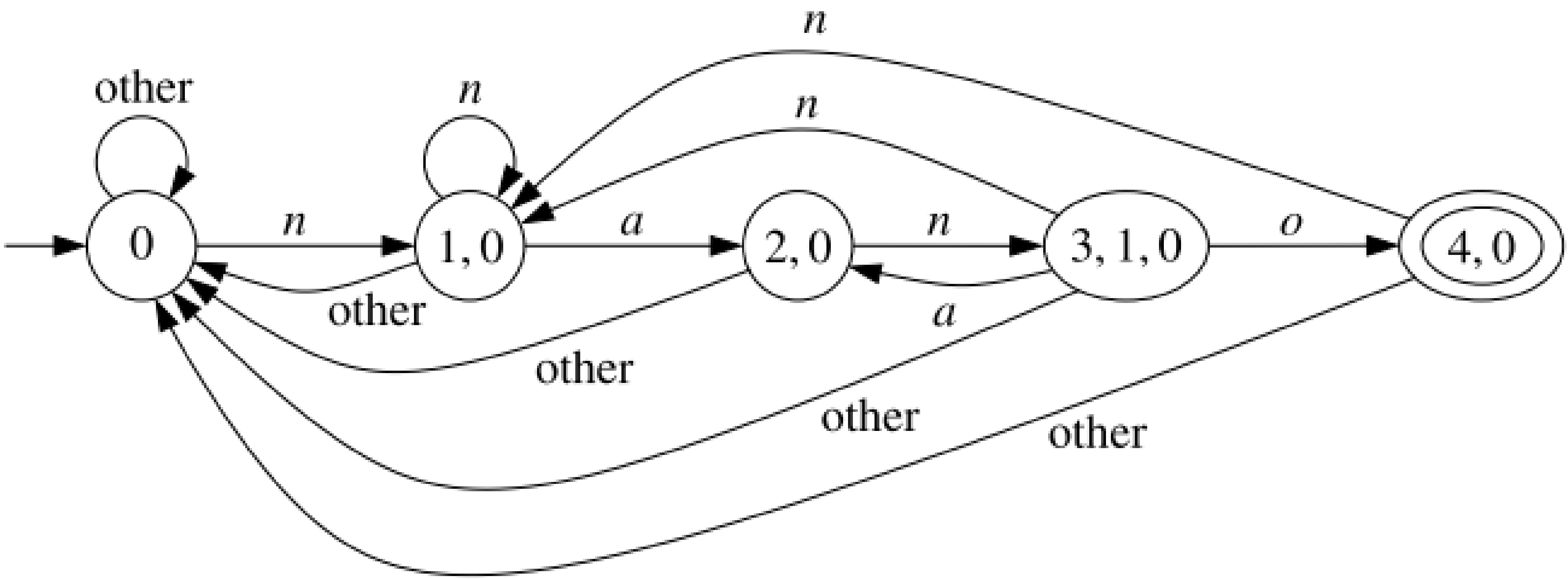
other
other

Intuition



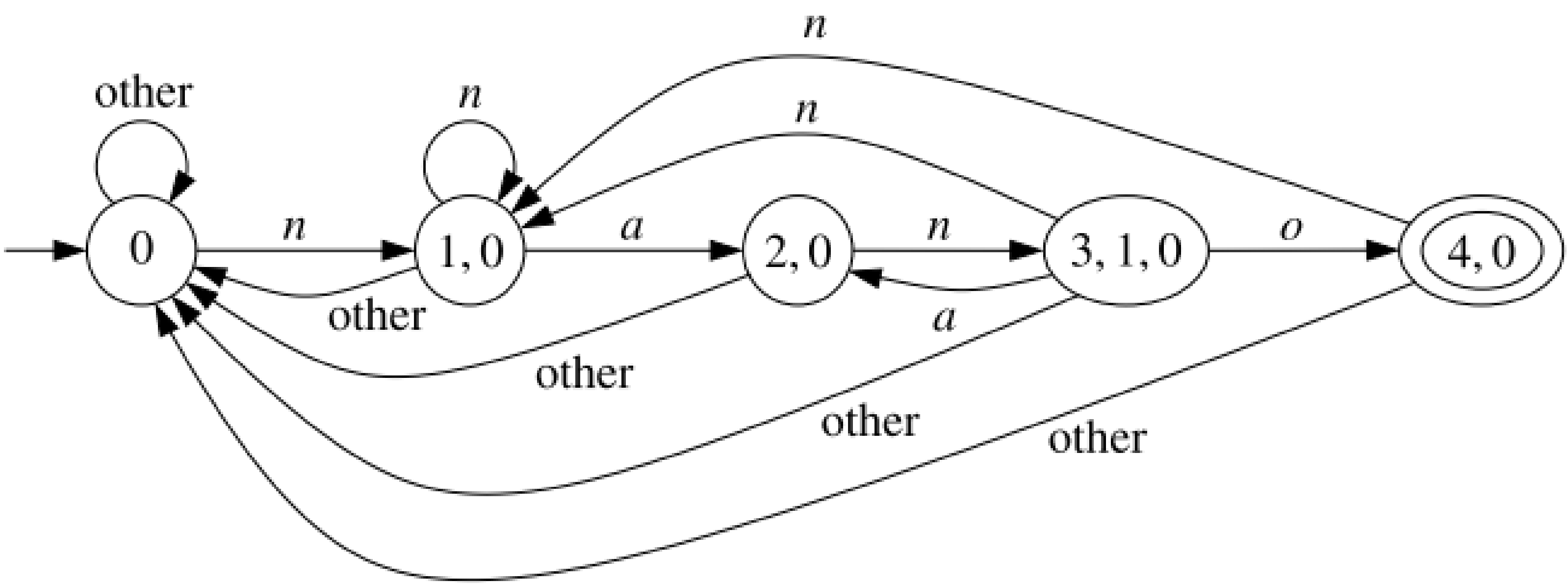
- Transitions of the „spine“ correspond to hits: the next letter is the one that „makes progress“ towards nano.
- Other transitions correspond to misses, i.e., „wrong letters“ and „throw the automaton back“.

Observations



- For every state $i = 0,1, \ldots, 4$ of the NFA there is exactly one state S of the DFA such that i is the largest state of S.
- For every state S of the DFA, with the exception of $S = \{0\}$, the result of removing the largest state is again a state of the DFA.

Observations



- For every state $i = 0, 1, \ldots, 4$ of the NFA there is exactly one state S of the DFA such that i is the largest state of S.
- For every state S of the DFA, with the exception of $S = \{0\}$, the result of removing the largest state is again a state of the DFA.
- Do these properties hold for every pattern p?

Heads and tails, hits and misses

- Head of S, denoted $h(S)$: largest state of S
- Tail of S, denoted $t(S)$: rest of the state
- Example: $h(\{3,1,0\}) = 3, \; t(\{3,1,0\}) = \{1,0\}$

- Given a state S, the letter leading to the next state in the „spine“ is the (unique) hit letter for S.
- All other letters are miss letters for S.
- Example: hit for $\{3,1,0\}$ is o, whereas n or a are misses.

Fundamental property of the DFA

- Proposition: Let S_k be the k-th state picked from the workset during the execution of *NFAtoDFA(A_p)*.

  (1) $h(S_k) = k$,

  (2) If $k > 0$, then $t(S_k) = S_l$ for some $l < k$

Proof Idea:

- (1) and (2) hold for $S_0 = \{0\}$ and $S_1 = \{1,0\}$.
- For the step $k \rightarrow k + 1$ we look at $\delta(S_k, a)$ for each a, where δ transition relation of A_p .
- By i.h. we have $S_k = \{k\} \cup S_l$ for some $l < k$.
- We distinguish two cases: a is a hit for S_k (that is, $a = b_{k+1}$), and a is a miss for S_k.

- $S_k = \{k\} \cup S_l$ for some $l < k$
- $\delta(S_k, a) = \delta(k, a) \cup \delta(S_l, a)$

Hit:

$$
\begin{array}{ccc}
\{k\} & \cup & S_l \\
\downarrow a & & \downarrow a \\
\{k+1\} & \cup & \delta(S_l, a)
\end{array}
$$

- $S_k = \{k\} \cup S_l$ for some $l < k$
- $\delta(S_k, a) = \delta(k, a) \cup \delta(S_l, a)$

$$\{k\} \quad \cup \quad S_l$$

Hit: a ↓ a ↓

$$\{k+1\} \cup \delta(S_l, a)$$

Added earlier to the workset, and so some $S_{l'}$

- $S_k = \{k\} \cup S_l$ for some $l < k$
- $\delta(S_k, a) = \delta(k, a) \cup \delta(S_l, a)$

$$\{k\} \quad \cup \quad S_l$$

Hit: a ↓ a ↓

$$\{k+1\} \cup \delta(S_l, a)$$

$$= \qquad\qquad =$$

$$\{k+1\} \cup S_{l'}$$

- $S_k = \{k\} \cup S_l$ for some $l < k$
- $\delta(S_k, a) = \delta(k, a) \cup \delta(S_l, a)$

$$\{k\} \quad \cup \quad S_l$$

Hit: $\quad a \downarrow \qquad\qquad a \downarrow$

$$\{k+1\} \cup \delta(S_l, a)$$

$$= \qquad\qquad =$$

$$\{k+1\} \cup S_{l'}$$

New state, gets added to the workset

- $S_k = \{k\} \cup S_l$ for some $l < k$
- $\delta(S_k, a) = \delta(k, a) \cup \delta(S_l, a)$

Hit:

$\{k\}$	$\cup$	S_l	$\{3\}$	$\cup$	$\{1,0\}$
a $\downarrow$		a $\downarrow$	o $\downarrow$		o $\downarrow$
$\{k+1\}$	$\cup$	$\delta(S_l, a)$	$\{4\}$	$\cup$	$\delta(\{1,0\}, o)$
=		=	=		=
$\{k+1\}$	$\cup$	$S_{l'}$	$\{4\}$	$\cup$	$\{0\}$

- $S_k = \{k\} \cup S_l$ for some $l < k$
- $\delta(S_k, a) = \delta(k, a) \cup \delta(S_l, a)$

$$
\begin{array}{ccc}
\{k\} & \cup & S_l \\
\downarrow a & & \downarrow a \\
\emptyset & \cup & \delta(S_l, a)
\end{array}
$$

Miss:

- $S_k = \{k\} \cup S_l$ for some $l < k$
- $\delta(S_k, a) = \delta(k, a) \cup \delta(S_l, a)$

Miss:

$$\{k\} \quad \cup \quad S_l$$

$$\downarrow a \qquad\qquad \downarrow a$$

$$\emptyset \quad \cup \quad \delta(S_l, a)$$

$$=$$

$$\color{red}{S_{l'}}$$

- $S_k = \{k\} \cup S_l$ for some $l < k$
- $\delta(S_k, a) = \delta(k, a) \cup \delta(S_l, a)$

$$\{k\} \quad \cup \quad S_l$$

Miss: a ↓ a ↓

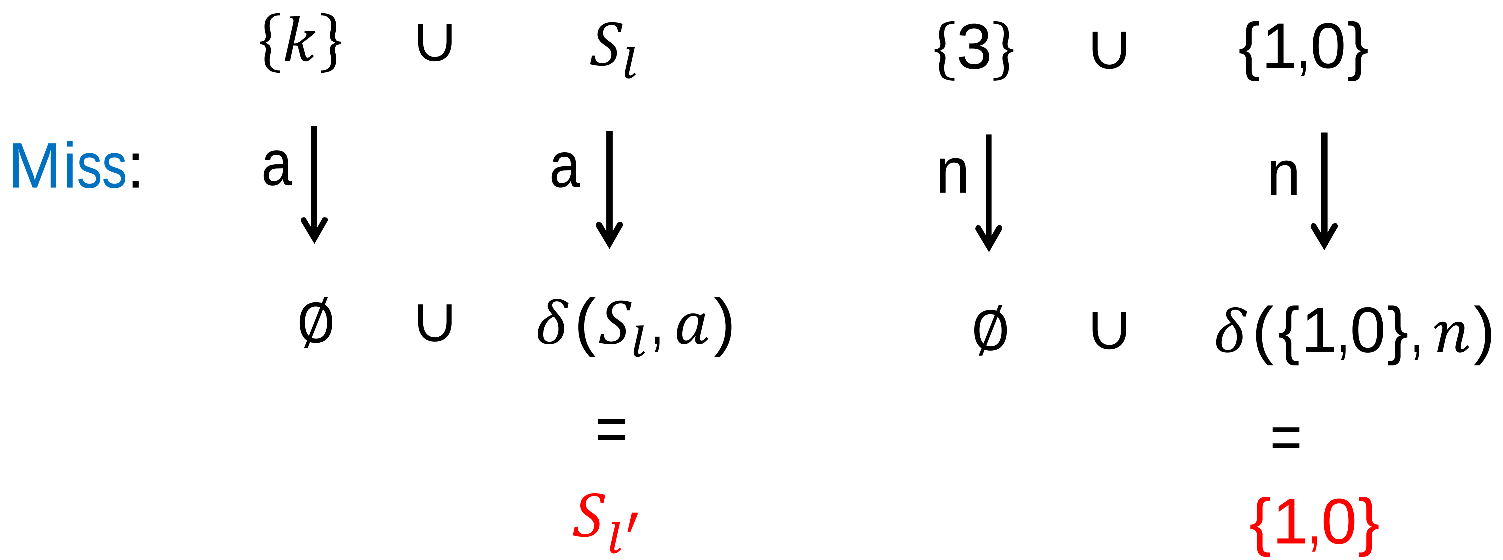
$$\emptyset \quad \cup \quad \delta(S_l, a)$$

$$=$$

$$S_{l'}$$

Already seen, is not added to the workset

- $S_k = \{k\} \cup S_l$ for some $l < k$
- $\delta(S_k, a) = \delta(k, a) \cup \delta(S_l, a)$



Consequences

Prop: The result of applying *NFAtoDFA*(A), where A is the obvious NFA for $\Sigma^* p$, yields a minimal DFA with $m + 1$ states and $|\Sigma|(m + 1)$ transitions.

Proof: All states of the DFA accept different languages.

So: concatenating *NFAtoDFA and PatternMatchingDFA* yields a $O(n + |\Sigma| m)$ algorithm.

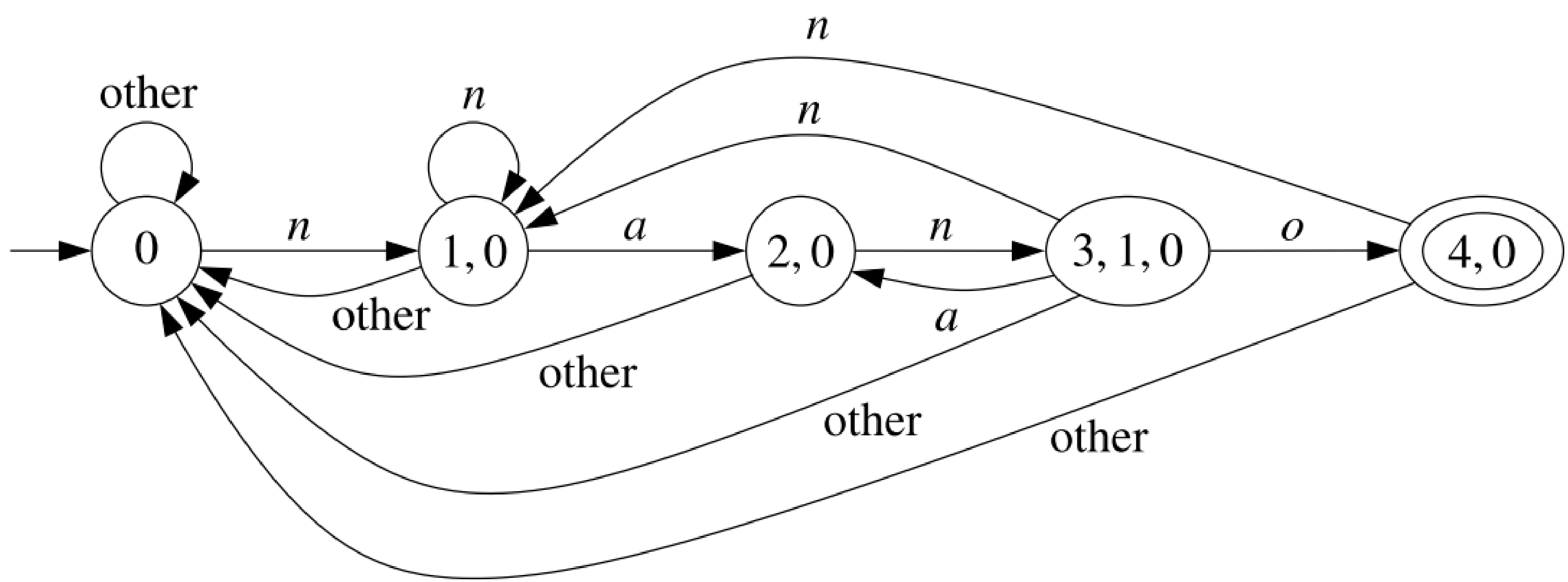
- Good enough for constant alphabet
- Not good enough for $|\Sigma| = \Omega(n)$, then same complexity as window algorithm

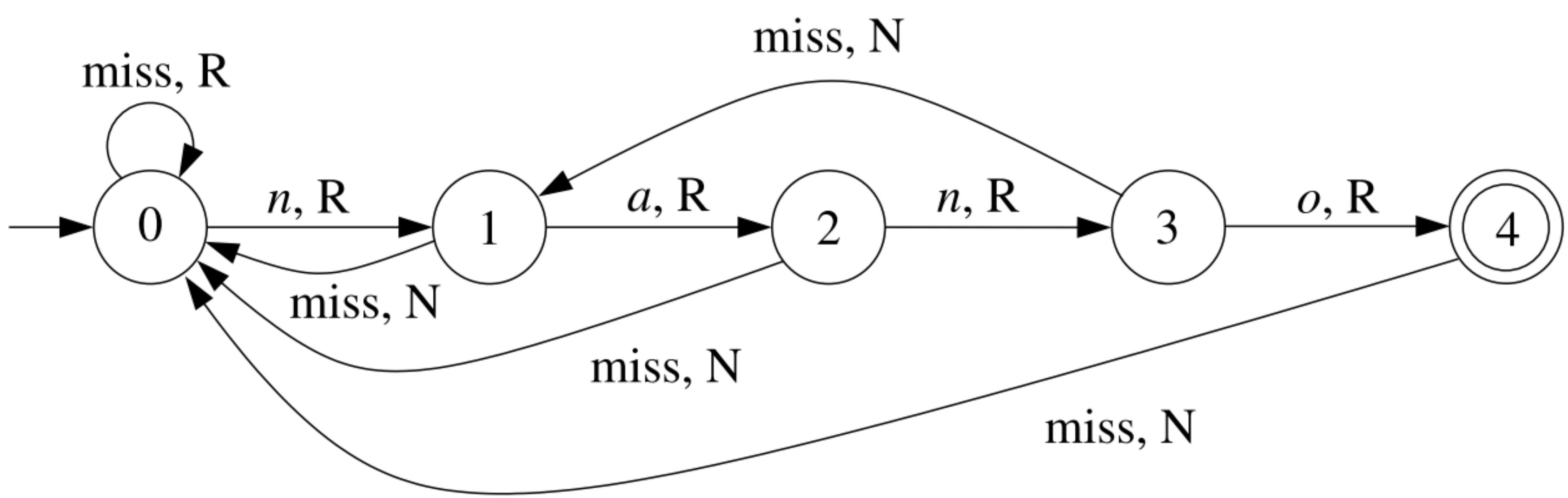
Lazy DFAs

- We introduce a new data structure: lazy DFAs. We construct a lazy DFA for $\Sigma^* p$ with $m + 1$ states and $2(m + 1)$ transitions.
- Lazy DFAs: automata that read the input from a tape by means of a reading head that can move one cell to the right or stay put.
- DFA = Lazy DFA whose head never stays put.

Lazy DFA for $\Sigma^* p$

- By the fundamental property, the DFA B for $\Sigma^* p$ behaves from state S_k as follows:
  - If a is a hit, then $\delta_B(S_k, a) = S_{k+1}$, i.e., the DFA moves to the next state in the spine.
  - If a is a miss, then $\delta_B(S_k, a) = \delta_B(t(S_k), a)$, i.e., the DFA moves to the same state it would move to if it were in state $t(S_k)$.
- When a is a miss for S_k, the lazy automaton moves to state $t(S_k)$ without advancing the head. In other words, it „delegates“ doing the move to $t(S_k)$.
- So the lazy DFA behaves the same for all misses.


other
n
n
n
0
1, 0
2, 0
3, 1, 0
4, 0
n
a
n
o
other
a
other
other
other


miss, N
miss, R
0
1
2
3
4
n, R
a, R
n, R
o, R
miss, N
miss, N
miss, N

- Formally, for the lazy DFA C:
  - $\delta_C(S_k, a) = (S_{k+1}, R)$ if a is a hit
  - $\delta_C(S_k, a) = (t(S_k), N)$ if a is a miss
- So the lazy DFA has $m + 1$ states and $2m$ transitions.
- It can be constructed in $O(m)$ space:
  - For each $0 \leq k \leq n$, compute and store S_k with
    - $S_0 \coloneqq \{0\}$, and
    - $S_{k+1} \coloneqq \delta(S_k, b_{k+1})$.
  - Compute the transitions as at the top of the slide.

- Running the lazy DFA on the text takes $O(n)$ time:
  - For every text letter the lazy DFA performs a sequence of „stay put“ steps followed by a „right“ step. Call this sequence a macrostep.
  - Let S_{j_i} be the state after the i-th macrostep. The number of steps of the i-th macrostep is at most $j_{i-1} - j_i + 2$.
  - So the total number of steps is at most

$$\sum_{i=1}^{n} (j_{i-1} - j_i + 2) = j_0 - j_n + 2n \leq 2n$$

Computing the lazy DFA in $O(m)$ time

- For the $O(m + n)$ bound it remains to show that the lazy DFA can be constructed in $O(m)$ time.
- Let $Miss(k)$ be the head of the state of the lazy DFA reached from S_k by a miss (that is, $Miss(k)$ is the head of the state $t(S_k)$).
- It is easy to compute each of $Miss(0), \ldots, Miss(m)$ in $O(m)$ time, leading to a $O(n + m^2)$ time algorithm.

  (Compute the S_k and use $Miss(k) = h(t(S_k))$.)
- Can we compute all of $Miss(0), \ldots, Miss(m)$ together in time $O(m)$? Looks impossible!
- It isn't though ...

Let $miss(i)$ be the state reached by a miss from S_i in the lazy DFA. Then $miss(i) = t(S_{\mathrm{i}})$ and $Miss(i) = h(miss(S_i))$.
For $i > 1$ we have

$$\begin{aligned} miss(i) = t(S_i) &= t(\ \delta_B(S_{i-1}, b_i)\ ) \\ &= t(\delta(\{i-1\}, b_i) \cup \delta(t(S_{i-1}), b_i)) \\ &= t\Big(\ \{i\} \cup \delta(t(S_{i-1}), b_i)\Big) \\ &= \delta_B(t(S_{i-1}), b_i) \end{aligned}$$

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For $i > 1$ we have

$$
\begin{aligned}
miss(i) = t(S_i) &= t(\delta_B(S_{i-1}, b_i)) \\
&= t(\delta(\{i-1\}, b_i) \cup \delta(t(S_{i-1}), b_i)) \\
&= t(\{i\} \cup \delta(t(S_{i-1}), b_i)) \\
&= \delta_B(t(S_{i-1}), b_i)
\end{aligned}
$$

and so we get

$$
miss(S_i) = \begin{cases} S_0 & \text{if } i = 0 \text{ or } i = 1 \\ \delta_B(miss(S_{i-1}), b_i) & \text{if } i > 1 \end{cases}
$$

$$
\delta_B(S_j, b) = \begin{cases} S_{j+1} & \text{if } b = b_{j+1} \text{ (hit)} \\ S_0 & \text{if } b \neq b_{j+1} \text{ (miss) and } j = 0 \\ \delta_B(miss(S_j), b) & \text{if } b \neq b_{j+1} \text{ (miss) and } j \neq 0 \end{cases}
$$

$$miss(S_i) = \begin{cases} S_0 & \text{if } i = 0 \text{ or } i = 1 \\ \delta_B(miss(S_{i-1}), b_i) & \text{if } i > 1 \end{cases}$$

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- With $Miss(i) = h(miss(S_i))$ we get the following algorithm:

CompMiss(p)
Input: pattern $p = b_1 \cdots b_m$.
Output: heads of targets of miss transitions.

```
Miss(0) ← 0; Miss(1) ← 0
for i ← 2, ..., m do
    Miss(i) ← DeltaB( Miss(i − 1) , b_i )
```

DeltaB(j, b)
Input: head $j \in \{0, \ldots, m\}$, letter b.
Output: head of the state $\delta_B(S_j, b)$.

```
while b ≠ b_{j+1} and j ≠ 0 do  j ← Miss(j)
if b = b_{j+1} then return j + 1
else return 0
```

CompMiss(p)
Input: pattern $p = b_1 \cdots b_m$.
Output: heads of targets of miss transitions.

1 $Miss(0) \leftarrow 0$; $Miss(1) \leftarrow 0$
2 **for** $i \leftarrow 2, \ldots, m$ **do**
3 $\quad Miss(i) \leftarrow DeltaB(\, Miss(i-1)\,,\, b_i\,)$

DeltaB(j, b)
Input: head $j \in \{0, \ldots, m\}$, letter b.
Output: head of the state $\delta_B(S_j, b)$.

1 **while** $b \neq b_{j+1}$ **and** $j \neq 0$ **do** $j \leftarrow Miss(j)$
2 **if** $b = b_{j+1}$ **then return** $j + 1$
3 **else return** 0

All calls to *DeltaB* lead together to $O(m)$ iterations of the while loop.
The call $DeltaB(Miss(i-1), b_i)$ executes at most

$$Miss(i-1) - (Miss(i) - 1)$$

iterations, because:

- initially j is assigned $Miss(i-1)$ (line 3 of *CompMiss*)
- each iteration decreases j by at least 1 (line 1 of $DeltaB$, $Miss(j) < j$)
- the return value assigned to $Miss(i)$ is at most the final value of j plus 1. (line 2 of $DeltaB$)

- Total number of iterations:

$$\sum_{i=2}^{m} (Miss(i-1) - Miss(i) + 1)$$

$$\leq \; Miss(1) - Miss(m) + m - 1$$

$$\leq m$$